

WELSCHINGER INVARIANTS OF BLOW-UPS OF SYMPLECTIC 4-MANIFOLDS

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ABSTRACT. Using the degeneration technique, one studies the behavior of Welschinger invariants under the blow-up, and obtains some blow-up formulae of Welschinger invariants. One also analyses the variation of Welschinger invariants when replacing a pair of real points in the real configuration by a pair of conjugated points, and reproves Welschinger's wall crossing formula.

1. INTRODUCTION

Traditional enumerative geometry asks certain question to which the expected answer is a number: for instance, the number of lines incident with two points in the plane, or the number of twisted cubic curves on a quintic 3-fold. For last two decades, the complex enumerative geometry of curves in algebraic varieties has taken a new direction with the appearance of Gromov-Witten invariants and quantum cohomology. The core of Gromov-Witten invariants is so called "counting the numbers of rational curves". On real enumerative geometry side, one expected a real version of Gromov-Witten invariants for long time. In 2005, Welschinger [36] first discovered such an invariant in dimensions 4 and 6, which was called the Welschinger invariant and revolutionized the real enumerative geometry. Recently it was partially extended to higher dimensions, higher genera and descendant type, see [8, 31, 32] and the references therein for the details. Itenberg-Kharlamov-Shustin [23] also extended the algebraic definition of Welschinger invariants to all del Pezzo surfaces and proved the invariance under deformation in algebraic setting.

After the Welschinger invariants were well defined for real symplectic 4-manifolds, the focus of the research on Welschinger invariants turned into its computation for some manifolds and the understanding of its global

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structure. Itenberg-Kharlamov-Shustin [15, 16, 18, 19, 21, 20] systematically studied the Welschinger invariants of del Pezzo surfaces, including the lower bounds of the invariants, the logarithmic equivalence of Welschinger and Gromov-Witten invariants, positivity, and Caporaso-Harris type formula for Welschinger invariants. Brugallé-Mikhalkin [5, 4] provided a method to compute Welschinger invariants via floor diagram. Using their method, Arroyo-Brugallé-de Medrano [1] computed the Welschinger invariants in the projective plane. Based on the open analogues of Kontsevich-Manin axioms and WDVV equation, Horev-Solomon [10] gave a recursive formula of Welschinger invariants of real blow up of the projective plane.

Using the degeneration technique, Itenberg-Kharlamov-Shustin [24] studied the positivity and asymptotics of Welschinger invariants of real del Pezzo surfaces of degree ≥ 2 , and obtained some new real Caporaso-Harris type formulae and real analogues of Abramovich-Bertram-Vakil formula. In [6, 7], Brugallé-Puignau applied the real version of symplectic sum formula to obtain a real version of Abramovich-Bertram-Vakil formula in the symplectic setting. Combining their formula with degeneration formula and the technique of floor diagrams relative to a conic, E. Brugallé [2] computed the Gromov-Witten invariants and Welschinger invariants of some del Pezzo surfaces.

Another important issues in the study of Welschinger invariants are how to understand the behavior of Welschinger invariants under geometric transformations and how to apply Welschinger invariants to investigate the geometry and topology of the underlying manifolds. In [2, 3, 7, 24], the authors used the degeneration technique to study the properties of Welschinger invariants. In particular, by locally modifying the real structure, E. Brugallé [3] proved very simple relations among Welschinger invariants of real symplectic 4-manifolds differing by a real surgery along a real Lagrangian sphere. In fact, his real surgery is a kind of real symplectic blow-up along a real Lagrangian sphere, see Section 4 of [3] for the details.

From the research of algebraic geometry and Gromov-Witten theory [26, 27], we know that the invariants obtained from the moduli spaces always have close relationship with the birational transformation. As well known, blow-up is the basic birational transformation. The moduli space of genus zero curves is well behaved under blow-up. The absolute value of Welschinger invariant provided a lower bound for the number of real pseudo-holomorphic curves passing through a particular real configuration and representing a degree, whereas an upper bound is given by the corresponding genus zero Gromov-Witten invariant. Inspired by the works on Gromov-Witten invariants [3, 11, 12, 13], we will study the behavior of Welschinger invariants under real symplectic blow-ups in this paper.

A *real symplectic 4-manifold* (X, ω, τ) , denoted by $X_{\mathbb{R}}$, is a symplectic 4-manifold (X, ω) with an involution τ on X such that $\tau^*\omega = -\omega$. The fixed point set of τ , denoted by $\mathbb{R}X$, is called the *real part* of X . $\mathbb{R}X$ is either empty or a smooth lagrangian submanifold of (X, ω) . An ω -tamed almost

complex structure J is called τ -compatible if τ is J -antiholomorphic. The space of all τ -compatible almost complex structures on X is denoted by $\mathbb{R}\mathcal{J}_\omega$. Let $c_1(X)$ be the first Chern class of the symplectic manifold (X, ω) . Let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Let L be a connected component of $\mathbb{R}X$. Assume $\underline{x} \subset X$ is a real configuration consisting of r real points in L and s pairs of τ -conjugated points in $X \setminus \mathbb{R}X$, where $r + 2s = c_1(X) \cdot d - 1$. Fix a τ -invariant class $F \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Denote by $W_{X_\mathbb{R}, L, F}(d, s)$ the Welschinger invariants. For the simplicity of notations, we assume that $\mathbb{R}X$ is connected and $F = 0$. In this situation, we denote $W_{X_\mathbb{R}}(d, s)$ instead of $W_{X_\mathbb{R}, L, F}(d, s)$.

Let $p : X_{a,b} \rightarrow X$ be the real symplectic blow-up of X at a real points and b pairs of τ -conjugated points. Denote by $p^!d = PDp^*PD(d)$, where PD stands for the Poincaré duality.

From the point of geometric view, an intuitive observation is that one will get the same number when we try to count the real rational pseudo-holomorphic curves in X and its blow-up in Welschinger's way if the blown-up points are away from the real configuration. This implies the following theorems.

Theorem 1.1. *Let $X_\mathbb{R}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \rightarrow X_\mathbb{R}$ the projection of the real symplectic blow-up of $X_\mathbb{R}$ at $x \in \mathbb{R}X$. Then*

$$(1) \quad W_{X_\mathbb{R}}(d, s) = W_{X_{1,0}}(p^!d, s),$$

$$(2) \quad W_{X_\mathbb{R}}(d, s) = W_{X_{1,0}}(p^!d - [E], s), \quad \text{if } c_1(X) \cdot d - 2s \geq 2,$$

where E denotes the exceptional divisor and $p^!d = PDp^*PD(d)$.

Theorem 1.2. *Let $X_\mathbb{R}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Suppose $y_1, y_2 \in X \setminus \mathbb{R}X$ is a τ -conjugated pair, i.e., $\tau(y_1) = y_2$. Denote by $p : X_{0,1} \rightarrow X_\mathbb{R}$ the projection of the real symplectic blow-up of $X_\mathbb{R}$ at y_1, y_2 . Then*

$$(3) \quad W_{X_\mathbb{R}}(d, s) = W_{X_{0,1}}(p^!d, s),$$

$$(4) \quad W_{X_\mathbb{R}}(d, s) = W_{X_{0,1}}(p^!d - [E_1] - [E_2], s - 1), \quad \text{if } s \geq 1,$$

where E_1, E_2 denote the exceptional divisors at y_1, y_2 respectively.

From Theorem 1.1 and Theorem 1.2, it is easy to get

Corollary 1.3. *Let $X_\mathbb{R}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Suppose $\underline{x}' \subset X$ is a real set consisting of r' points in $\mathbb{R}X$ and s' pairs of τ -conjugated points in $X \setminus \mathbb{R}X$ with $r' \leq r$, $s' \leq s$. Denote by $p : X_{r',s'} \rightarrow X$ the projection of the real symplectic blow-up of X at \underline{x}' . Then*

$$(5) \quad W_{X_\mathbb{R}}(d, s) = W_{X_{r',s'}}(p^!d, s),$$

$$(6) \quad W_{X_{\mathbb{R}}}(d, s) = W_{X_{r', s'}}(p^!d - \sum_{i=1}^{r'} [E_i] - \sum_{j=1}^{s'} ([E'_j] + [E''_j]), s - s'),$$

where E_i, E'_j, E''_j denote the exceptional divisors corresponding to the real set \underline{x}' respectively.

Welschinger [36] introduced a new θ -invariant to describe the dependence of Welschinger invariants on the number of real points in the real configurations, and obtained a **wall-crossing formula**, see Theorem 3.2 of [36]. More precisely, when replacing a pair of real fixed points in the same component of $\mathbb{R}X$ by a pair of imaginary conjugate points, twice the θ -invariant is the difference of the respective invariants. In this paper, using the degeneration method, we reprove Welschinger's wall-crossing formula and verify that Welschinger's θ -invariants of $X_{\mathbb{R}}$ are the Welschinger invariants of the real blow-up $X_{1,0}$ of the real symplectic manifold at one real point.

Theorem 1.4. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d \geq 4$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \rightarrow X$ the projection of the real symplectic blow-up of $X_{\mathbb{R}}$ at $x \in \mathbb{R}X$. If $s \geq 1$, then*

$$(7) \quad W_{X_{\mathbb{R}}}(d, s-1) = W_{X_{\mathbb{R}}}(d, s) + 2W_{X_{1,0}}(p^!d - 2[E], s-1),$$

where E denotes the exceptional divisor and $p^!d = PDp^*PD(d)$.

Remark 1.5. The same argument as in the proofs of previous theorems generalizes the formulae (5) and (6) to the general case that $\mathbb{R}X$ is disconnected. More precisely, assume that $X_{\mathbb{R}}$ is a compact real symplectic 4-manifold and $\mathbb{R}X$ is disconnected. Suppose $\underline{x}' \subset X$ is a real set made of r' points in L and s' pairs of τ -conjugated points in X with $r' \leq r, s' \leq s$. Denote by \tilde{L} the connected component of $\mathbb{R}X_{r', s'}$ corresponding to L . If only one of the blown-up real points belongs to L , $\tilde{L} = L \# \mathbb{R}P^2$. We assume F has a τ -invariant compact representative $\mathcal{F} \subset X \setminus \underline{x}'$, and denote $\tilde{F} = p^!F$. Denote by $p : X_{r', s'} \rightarrow X$ the projection of the real symplectic blow-up of X at \underline{x}' . Then

$$(8) \quad W_{X_{\mathbb{R}}, L, F}(d, s) = W_{X_{r', s'}, \tilde{L}, \tilde{F}}(p^!d, s),$$

$$(9) \quad W_{X_{\mathbb{R}}, L, F}(d, s) = W_{X_{r', s'}, \tilde{L}, \tilde{F}}(p^!d - \sum_{i=1}^{r'} [E_i] - \sum_{j=1}^{s'} ([E'_j] + [E''_j]), s - s'),$$

$$(10) \quad W_{X_{\mathbb{R}}, L, F}(d, s-1) = W_{X_{\mathbb{R}}, L, F}(d, s) + 2W_{X_{1,0}, \tilde{L}, \tilde{F}}(p^!d - 2[E], s-1),$$

where E_i, E'_j, E''_j denote the exceptional divisors corresponding to the real set \underline{x}' respectively.

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2. PRELIMINARY

2.1. Real blow-ups of the projective plane. In this subsection, we consider how the standard real structure, i. e., the conjugate on \mathbb{CP}^2 induces a real structure on $\tilde{\mathbb{CP}}^2$. For this purpose, we must distinguish the real points from other points in $\mathbb{CP}^2 \setminus \mathbb{RP}^2$.

First of all, we review the blow-up of \mathbb{CP}^2 at a point x . Let U be a neighborhood of x with local coordinate (z_1, z_2) . Denote

$$\pi : V := \{((z_1, z_2), [w_1 : w_2]) \in U \times \mathbb{CP}^1 \mid z_i w_j = z_j w_i\} \longrightarrow U$$

the projection to U via the first factor. There is a natural identification map gl between $V \setminus E = \pi^{-1}(0)$ and $U \setminus \{x\}$, where $E = \pi^{-1}(0)$ is the exceptional divisor. We get the blow-up

$$\tilde{\mathbb{CP}}^2 = \mathbb{CP}^2 \setminus \{x\} \bigcup_{gl} V.$$

For a real point $x \in \mathbb{RP}^2 \subset \mathbb{CP}^2$, denote by $\tilde{\mathbb{CP}}_{1,0}^2$ the blow-up of \mathbb{CP}^2 at x . We may choose a conjugation invariant neighborhood U of x in \mathbb{CP}^2 and the local coordinate (z_1, z_2) .

Define an involution $\tau : V \longrightarrow V$ as

$$\tau(((z_1, z_2), [w_1 : w_2])) := ((\bar{z}_1, \bar{z}_2), [\bar{w}_1 : \bar{w}_2]).$$

It is easy to check that this involution coincides with the one induced on $V \setminus E$ by identification with $U \setminus \{x\}$. This implies that the standard real structure on \mathbb{CP}^2 naturally induces a real structure on $\tilde{\mathbb{CP}}_{1,0}^2$ at a real point $x \in \mathbb{RP}^2$.

Since there is no conjugation-invariant neighborhood for the points in $\mathbb{CP}^2 \setminus \mathbb{RP}^2$, to obtain a real structure on the blow-up, we need to blow up simultaneously a pair of two conjugated points. Let x_1 and $x_2 = \bar{x}_1$ be a pair of conjugated points in \mathbb{CP}^2 . Choose a neighborhood U_1 of x_1 with local coordinate (z_1, z_2) , a neighborhood $U_2 = \bar{U}_1$ of x_2 with local coordinate (\bar{z}_1, \bar{z}_2) . Denote by $\tilde{\mathbb{CP}}_{0,1}^2$ the blow-up of \mathbb{CP}^2 at points x_1 and x_2 and π the natural projection of the blow-up. Denote

$$V_1 := \{((z_1, z_2), [w_1 : w_2]) \in U_1 \times \mathbb{CP}^1 \mid z_i w_j = z_j w_i\} = \pi^{-1}(U_1),$$

$$V_2 := \{((z_1, z_2), [w_1 : w_2]) \in U_2 \times \mathbb{CP}^1 \mid z_i w_j = z_j w_i\} = \pi^{-1}(U_2).$$

Define the involution $\tau : V_1 \cup V_2 \longrightarrow V_1 \cup V_2$ as

$$\tau(((z_1, z_2), [w_1 : w_2])) := ((\bar{z}_1, \bar{z}_2), [\bar{w}_1 : \bar{w}_2]).$$

This involution coincides by construction with the one induced by the identification $\mathbb{CP}^2 \setminus \{x_1, x_2\} = \tilde{\mathbb{CP}}_{0,1}^2 \setminus \{\pi^{-1}(x_1), \pi^{-1}(x_2)\}$. Therefore, we also can obtain a natural real structure on the blow-up $\tilde{\mathbb{CP}}_{0,1}^2$.

2.2. Symplectic cut. Lerman's symplectic cutting [25] is a simple and versatile operation on Hamiltonian S^1 -manifolds. Suppose that $X_0 \subset X$ is an open codimension zero connected submanifold with a Hamiltonian S^1 -action. Let $H : X_0 \rightarrow \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a separating hypersurface of X , then we obtain two connected manifolds X_0^\pm with boundary $\partial X_0^\pm = H^{-1}(0)$, where the $+$ side corresponds to $H < 0$. Suppose further that S^1 acts freely on $H^{-1}(0)$. Then the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold of dimension 2 less. Collapsing the S^1 -action on $\partial X^\pm = H^{-1}(0)$, we obtain smooth closed manifolds \bar{X}^\pm containing respectively real codimension 2 submanifolds $Z^\pm = Z$ with opposite normal bundles. Furthermore \bar{X}^\pm admits a symplectic structure $\bar{\omega}^\pm$ which agrees with the restriction of ω away from Z , and whose restriction to Z^\pm agrees with the canonical symplectic structure ω_Z on Z from symplectic reduction. The pair of symplectic manifolds $(\bar{X}^\pm, \bar{\omega}^\pm)$ is called the **symplectic cut** of X along $H^{-1}(0)$.

This is neatly shown by considering $X_0 \times \mathbb{C}$ equipped with appropriate product symplectic structures and the product S^1 -action on $X_0 \times \mathbb{C}$ where S^1 acts on \mathbb{C} by complex multiplication. The extended action is Hamiltonian if we use the standard symplectic structure $\sqrt{-1}dw \wedge d\bar{w}$ or its negative on the \mathbb{C} factor, see [25]. Denote by $\mu : X_0 \rightarrow \mathbb{R}$ the moment map of the S^1 -action on X_0 , then $\bar{X}^+ = \bar{X}_{\mu \leq \epsilon}$, $\bar{X}^- = \bar{X}_{\mu \geq \epsilon}$.

The normal connected sum operation [9, 28] or the fiber sum operation is the inverse operation of the symplectic cut. Given two symplectic manifolds containing symplectomorphic codimension 2 symplectic submanifolds with opposite normal bundles, the normal connected sum operation produces a new symplectic manifold by identifying the tubular neighborhoods.

Notice that we can apply the normal connected sum operation to the pairs $(\bar{X}^+, \bar{\omega}^+, Z^+)$ and $(\bar{X}^-, \bar{\omega}^-, Z^-)$ to recover (X, ω) .

According to McDuff [29], the blow-up operation in symplectic geometry amounts to a removal of an open symplectic ball followed by a collapse of some boundary directions. In fact, we may apply the symplectic cut to construct the blow-up of symplectic manifold X at a point p . By the symplectic neighborhood theorem, take X_0 be a symplectic ball of radius ϵ_0 centered at p with complex coordinates (z_1, \dots, z_n) , where $\dim X = 2n$. Consider the Hamiltonian S^1 -action on X_0 by complex multiplication. Fix ϵ with $0 < \epsilon < \epsilon_0$ and consider the moment map

$$H(u) = \sum_{i=1}^n |z_i|^2 - \epsilon, \quad u \in X_0.$$

Write the hypersurface $P = H^{-1}(0)$ in X corresponding to the sphere with radius ϵ . We cut X along P to obtain two closed symplectic manifolds \bar{X}^+ and \bar{X}^- , one of which is $\mathbb{C}P^n$. Following the notations of [11, 26], we denote $\bar{X}^+ = \mathbb{C}P^n$, $\bar{X}^- = \tilde{X}$ is the **symplectic blow-up** of X .

2.3. Real symplectic cut. Let (X, ω_X, τ_X) and (Y, ω_Y, τ_Y) be two real compact symplectic manifolds. Suppose $V \subset X$ and $V \subset Y$ are real symplectic hypersurfaces so that $e(N_{V|X}) + e(N_{V|Y}) = 0$, where $N_{V|X}$, $N_{V|Y}$ are the normal bundles of V in X and Y respectively. Denote by ω_V the symplectic form $\omega_X|_V = \omega_Y|_V$ on V and by τ_V the real structure $\tau_X|_V = \tau_Y|_V$. There is a real structure τ_{\sharp} on $X \sharp_V Y$ induced by the real structures τ_X , τ_Y . Suppose $\pi : \mathcal{Z} \rightarrow \Delta$ is the symplectic sum (c.f. [9, 14, 28]). Equip the disc Δ with the complex conjugate. There is a real structure $\tau_{\mathcal{Z}}$ on \mathcal{Z} induced by τ_X and τ_Y such that the map $\pi : \mathcal{Z} \rightarrow \Delta$ is real. See [3] or [7] for more about real symplectic sum in dimension 4.

Let (X, ω, τ) be a real symplectic manifold. Assume $H : X \rightarrow \mathbb{R}$ is a τ -invariant smooth Hamiltonian i.e., $H \circ \tau = H$, then we call H a *real Hamiltonian* (c.f. [34]). A Hamiltonian circle action on (X, ω, τ) is a 1-parameter subgroup $\mathbb{R} \rightarrow \text{Symp}(X) : t \mapsto \psi_t$ of symplectomorphisms of X which is 2π -periodic, i.e. $\psi_{2\pi} = \text{id}$, and which is the integral of a Hamiltonian vector field X_H . The Hamiltonian function $H : X \rightarrow \mathbb{R}$ in this case is called the moment map of the action. If the Hamiltonian circle action on (X, ω, τ) satisfies

$$(11) \quad \psi_{2\pi-t} \circ \tau = \tau \circ \psi_t$$

for all $t \in [0, 2\pi]$, we call it a *real Hamiltonian circle action*. The moment map of a real Hamiltonian circle action is a real Hamiltonian.

Let (X, ω, τ) be a real symplectic manifold with a real Hamiltonian circle action. Suppose that $\mu : X \rightarrow \mathbb{R}$ is a real moment map. Let $(\mu^{-1}(0)/S^1, \omega_{\mu})$ be the symplectic reduction. There is a natural real structure τ_{μ} on $\mu^{-1}(0)/S^1$ induced by τ on X . Define

$$\begin{aligned} \tau_{\mu} : \mu^{-1}(0)/S^1 &\rightarrow \mu^{-1}(0)/S^1 \\ [x] &\mapsto [\tau(x)]. \end{aligned}$$

Suppose $x, y \in \mu^{-1}(0)$ such that $[x] = [y]$, then there is a $t \in [0, 2\pi]$ such that $\psi_t(x) = y$. By equation (11), $\psi_{2\pi-t}(\tau(x)) = \tau \circ \psi_t(x) = \tau(y)$. Therefore, $[\tau(x)] = [\tau(y)]$ and τ_{μ} is well defined. Obviously, the reduced space $(\mu^{-1}(0)/S^1, \omega_{\mu}, \tau_{\mu})$ is a real symplectic manifold.

Then we can find that the real symplectic manifold $(\mu^{-1}(\varepsilon)/S^1, \omega_{\mu}, \tau_{\mu})$ is a real symplectic manifold embedded in both $\bar{X}_{\mu \geq \varepsilon}$ and $\bar{X}_{\mu \leq \varepsilon}$ as a codimension 2 real symplectic submanifold but with opposite normal bundles. The pair of real symplectic manifolds $\bar{X}_{\mu \geq \varepsilon}$, $\bar{X}_{\mu \leq \varepsilon}$ is called the **real symplectic cut** of X along $\mu = \varepsilon$.

Remark 2.1. Let $X_0 \subset X$ be an open codimension zero real submanifold equipped with real Hamiltonian S^1 action with a real proper momentum map $\mu : X_0 \rightarrow \mathbb{R}$. Suppose that μ achieves its maximal value c at a single point $p \in \mathbb{R}X$. For a sufficient small ε , p is the only critical point in the set $X_{\mu > c-\varepsilon} = \{x \in X_0 | c - \mu(x) < \varepsilon\}$. For all $0 < \delta < \varepsilon$, the real symplectic manifold $\bar{X}_{\mu \leq c-\delta}$ is the real blow-up of X_0 at p by a δ amount. We define

$\bar{X}^+ := \bar{X}_{\mu \geq c-\delta}$ and $\bar{X}^- := (X - X_0) \cup \bar{X}_{\mu \leq c-\delta}$. We have $\bar{X}^+ = \mathbb{C}P^n$, $\bar{X}^- = X_{1,0}$, where $X_{1,0}$ is the real blow-up of X at a real point. Suppose that μ achieves its maximal value c at a single τ -conjugated pair $p_1, p_2 \in X \setminus \mathbb{R}X$. Then $\bar{X}^+ = \mathbb{C}P^n \amalg \mathbb{C}P^n$, $\bar{X}^- = X_{0,1}$, where $X_{0,1}$ is the real blow-up of X at a pair of τ -conjugated points.

2.4. Welschinger invariants. Let (X, ω, τ) be a compact real symplectic 4-manifold. Assume that the first Chern class $c_1(X)$ of the symplectic manifold (X, ω) is not a torsion element and let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Let $\underline{x} = (x_1, \dots, x_m)$ be an ordered set of distinct points of X such that $\tau(\underline{x}) = \underline{x}$. Such a set is called a *real configuration* of points. Let $\sigma(\tau)$ be the order two permutation of $\{1, \dots, m\}$ induced by τ . Let S be an oriented 2-sphere, \mathcal{J}_S be the space of complex structure of class C^l of S compatible with its orientation and $\underline{z} = (z_1, \dots, z_m)$ be m distinct points on S . Define $\mathcal{P}^d(\underline{x})$ be the set of pseudo-holomorphic maps from S to X which pass through \underline{x} and represent class d . Let $\mathcal{P}^*(\underline{x})$ be the subspace of $\mathcal{P}^d(\underline{x})$ consisting of simple maps.

Denote by $\text{Diff}(S, \underline{z})$ the group of diffeomorphisms of class C^{l+1} of S , which either preserve the orientation and fix \underline{z} , or reverse the orientation and induce the permutation on \underline{z} associated to τ . Let $\text{Diff}^+(S, \underline{z})$ (resp. $\text{Diff}^-(S, \underline{z})$) be the subgroup of $\text{Diff}(S, \underline{z})$ of orientation preserving diffeomorphisms (resp. its complement in $\text{Diff}(S, \underline{z})$). The group $\text{Diff}(S, \underline{z})$ acts on $\mathcal{P}^d(\underline{x}) \times \mathcal{J}_S \times \mathcal{J}_\omega$ by

$$\phi(u, J_S, J) = \begin{cases} (u \circ \phi^{-1}, (\phi^{-1})^* J_S, J), & \text{if } \phi \in \text{Diff}^+(S, \underline{z}), \\ (\tau \circ u \circ \phi^{-1}, (\phi^{-1})^* J_S, \tau^* J), & \text{if } \phi \in \text{Diff}^-(S, \underline{z}), \end{cases}$$

Denote by $\mathcal{M}^d(\underline{x})$ the quotient of $\mathcal{P}^d(\underline{x})$ by the action of $\text{Diff}^+(S, \underline{z})$. Let $\pi : \mathcal{M}^d(\underline{x}) \rightarrow \mathcal{J}_\omega$ be the projection.

Proposition 2.2. [36, Proposition 1.8] *The space $\mathcal{M}^d(\underline{x})$ is a separable Banach manifold of class C^{l-k} . The projection $\pi : \mathcal{M}^d(\underline{x}) \rightarrow \mathcal{J}_\omega$ is Fredholm of index $\text{Ind}_{\mathbb{R}}(\pi) = 2(c_1(X) \cdot d - 1 - m)$.*

The manifold $\mathcal{M}^d(\underline{x})$ is equipped with an $\mathbb{Z}/2\mathbb{Z}$ action. Let $\mathbb{R}\mathcal{M}^d(\underline{x})$ denote the fixed point set of this action. $\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(\underline{x}) \rightarrow \mathbb{R}\mathcal{J}_\omega$ is the projection.

Proposition 2.3. [36, Proposition 1.9] *The projection $\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(\underline{x}) \rightarrow \mathbb{R}\mathcal{J}_\omega$ is Fredholm of index $\text{Ind}_{\mathbb{R}}(\pi_{\mathbb{R}}) = c_1(X) \cdot d - 1 - m$.*

Suppose $\mathbb{R}X$ is connected. Let $c_1(X) \cdot d - 1 = r + 2s$, $\underline{x} \subset X$ be a real configuration consisting of r real points in $\mathbb{R}X$ and s pairs of τ -conjugated points in $X \setminus \mathbb{R}X$. In this case, we denote $\mathcal{C}(d, \underline{x}, J) = \mathcal{M}^d(\underline{x})$, $\mathbb{R}\mathcal{C}(d, \underline{x}, J) = \mathbb{R}\mathcal{M}^d(\underline{x})$.

Proposition 2.4. [36] *Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic, the set $\mathcal{C}(d, \underline{x}, J)$ is finite. Moreover, these curves are all irreducible and have only transversal double*

points as singularities. The total number of double points of curve C in $\mathcal{C}(d, \underline{x}, J)$ equal to $\delta = \frac{1}{2}(d^2 - c_1(X) \cdot d + 2)$.

Assume $C \in \mathbb{R}\mathcal{M}^d(\underline{x})$. The real double points of C are of two different kinds. They are either non-isolated or isolated. A real double point is called *non-isolated* if it is the local intersection of two real branches. The real nodal point which is the local intersection of two complex conjugated branches is called *isolated*. The *mass* $m_X(C)$ is defined to be the number of its isolated real nodal points which satisfies $0 \leq m(C) \leq \delta$. The integer

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C \in \mathbb{R}\mathcal{C}(d, \underline{x}, J)} (-1)^{m_X(C)}$$

neither depends on the choice of J, \underline{x} , nor on the deformation class of $X_{\mathbb{R}}$ (c.f.[35, 36]). These numbers are called *Welschinger invariants* of $X_{\mathbb{R}}$.

When the real part $\mathbb{R}X$ is disconnected, let L be a connected component of $\mathbb{R}X$. Suppose $f : S \rightarrow X$ is an immersed real rational J -holomorphic curve in X such that $f(\mathbb{R}S) \subset L$, for a $J \in \mathbb{R}\mathcal{J}_{\omega}$. Denoting by S^+ a half of $S \setminus \mathbb{R}S$, $f(S^+)$ defines a class $[f(S^+)]$ in $H_2(X, L; \mathbb{Z}/2\mathbb{Z})$. There exists a well defined pairing

$$H_2(X, L; \mathbb{Z}/2\mathbb{Z}) \times H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$$

given by the intersection product modulo 2. Fix a τ -invariant class $F \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Define the (L, F) -mass of f as

$$m_{L,F}(f) = m_L(f) + [f(S^+)] \cdot F,$$

where $m_L(f)$ is the number of real isolated nodes of f in L . $m_{L,F}(f)$ does not depend on the chosen half of $S \setminus \mathbb{R}S$.

Given $J \in \mathbb{R}\mathcal{J}_{\omega}$, the set $\mathbb{R}\mathcal{C}(d, \underline{x}, J)$ consists of real rational J -holomorphic curves $f : S \rightarrow X$ in X realizing the class d , passing through \underline{x} , and such that $f(\mathbb{R}S) \subset L$. Note that if $r \geq 1$, the condition $f(\mathbb{R}S) \subset L$ is always satisfied. Itenberg, Kharlamov and Shustin [20] observed that the integer

$$W_{X_{\mathbb{R}}, L, F}(d, s) = \sum_{C \in \mathbb{R}\mathcal{C}(d, \underline{x}, J)} (-1)^{m_{L,F}(C)}$$

neither depends on the choice of J, \underline{x} , nor on the deformation class of $X_{\mathbb{R}}$. Note that if $F = [\mathbb{R}X \setminus L]$, $W_{X_{\mathbb{R}}, L, F}(d, s)$ is the original Welschinger invariant. For the simplicity of notation, we assume $\mathbb{R}X$ is connected and $F = 0$. At this situation, we denote $W_{X_{\mathbb{R}}}(d, s)$ instead of $W_{X_{\mathbb{R}}, L, F}(d, s)$.

2.5. Curves with tangency conditions. When we use the degeneration technique to study the behavior of curves under blow-up, we need to deal with curves with tangency conditions. In this subsection, we review some basics on curves with tangency conditions. We use Section 2.1 of [7] as the reference.

Two J -holomorphic maps $f_1 : C_1 \rightarrow X$ and $f_2 : C_2 \rightarrow X$ are said to be isomorphism if there is a biholomorphism $\phi : C_1 \rightarrow C_2$ such that $f_1 = f_2 \circ \phi$.

In the following, maps are always considered up to isomorphism. Given a vector $\alpha = (\alpha_i)_{1 \leq i < \infty} \in \mathbb{Z}_{\geq 0}^\infty$, we use the notation:

$$|\alpha| = \sum_{i=1}^{+\infty} \alpha_i, \quad I\alpha = \sum_{i=1}^{+\infty} i\alpha_i.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i)_{1 \leq i < \infty}$, denote $k\alpha := (k\alpha_i)_{1 \leq i < \infty}$. Let δ_i denote the vector in $\mathbb{Z}_{\geq 0}^\infty$ whose all coordinates are equal to 0 except the i th one which is equal to 1.

Let (X, ω) be a compact and connected 4-dimensional symplectic manifold, and let $V \subset X$ be an embedded symplectic curve in X . Let $d \in H_2(X; \mathbb{Z})$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}^\infty$ such that

$$I\alpha + I\beta = d \cdot [V].$$

Choose a configuration $\underline{x} = \underline{x}^\circ \sqcup \underline{x}_V$ of points in X , with \underline{x}° a configuration of $c_1(X) \cdot d - 1 - d \cdot [V] + |\beta|$ points in $X \setminus V$, and $\underline{x}_V = \{p_{i,j}\}_{0 < j \leq \alpha_i, i \geq 1}$ a configuration of $|\alpha|$ points in V . Given an ω -tamed almost complex structure J on X such that V is J -holomorphic, denote by $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ the set of rational J -holomorphic maps $f : \mathbb{C}P^1 \rightarrow X$ such that

- $f_*[\mathbb{C}P^1] = d$;
- $\underline{x} \subset f(\mathbb{C}P^1)$;
- V does not contain $f(\mathbb{C}P^1)$;
- $f(\mathbb{C}P^1)$ has order of contact i with V at each points $p_{i,j}$;
- $f(\mathbb{C}P^1)$ has order of contact i with V at exactly β_i distinct points on $V \setminus \underline{x}_V$.

The set of simple maps in $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is 0-dimensional if the almost complex structure J is chosen to be generic. However, $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ might contain components of positive dimension corresponding to non-simple maps.

Lemma 2.5. [7, Lemma 11] *Suppose that $\beta = (d \cdot [V], 0, \dots)$ and $\alpha = 0$, or $\beta = (d \cdot [V] - 1, 0, \dots)$ and $\alpha = (1, 0, \dots)$. Then for a generic choice of J , the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ only contains simple maps.*

Proposition 2.6. [7, Proposition 13] *Suppose that V is an embedded symplectic sphere with $[V]^2 = -1$, and that $|\beta| \geq d \cdot [V] - 1$. Then for a generic choice of J , the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ contains finitely many simple maps. As a consequence, the set*

$$\mathcal{C}_*^{\alpha,\beta}(d, \underline{x}, J) = \{f(\mathbb{C}P^1) | (f : \mathbb{C}P^1 \rightarrow X) \in \mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)\}$$

is also finite.

In particular, suppose that $X = \mathbb{C}P^2$, $V = H \subset \mathbb{C}P^2$ is the hyperplane in $\mathbb{C}P^2$, and $|\underline{x}^0| = 1$, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is always finite and made of simple maps.

Lemma 2.7. *Suppose that $X = \mathbb{C}P^2$ and $V = H \subset \mathbb{C}P^2$ is the hyperplane in $\mathbb{C}P^2$. Then the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ with $|\underline{x}^0| = 1$ is empty for a generic*

choice of J , except $\mathcal{C}^{\delta_1,0}([H], \{p\} \cup \underline{x}_V, J)$ which contains an unique element. Moreover, this unique element is an embedding.

Proof. Suppose that $d = a[H]$, $a \geq 0$. Since $c_1(X) = 3[H]$, we have

$$c_1(X) \cdot (a[H]) - 1 - (a[H]) \cdot H + |\beta| = 2a - 1 + |\beta|.$$

Suppose that $2a - 1 + |\beta| = 1$ and $\mathcal{C}^{\alpha,\beta}(a[H], \underline{x}, J) \neq \emptyset$, where $|\underline{x}^0| = 1$. From $I\alpha + I\beta = d \cdot [V] = a[H] \cdot [V] = a$, we can get $a[H] \cdot [H] = a \geq |\beta|$. The intersection number a of J -holomorphic curve $f \in \mathcal{C}^{\alpha,\beta}(a[H], \underline{x}, J)$ with V must satisfy $a \geq 0$. So we obtain $a = 1$, $|\beta| = 0$.

If $\mathbb{C}P^2$ is equipped with the symplectic form ω_{FS} and its standard complex structure J_{st} , it is well-known that $\mathcal{C}^{\delta_1,0}([H], \{p\} \cup \underline{x}_V, J_{st})$ consists of a unique element. When ω and J are both varied, the corresponding set still contains at least one element. If there are two distinct curves C_1 and C_2 in $\mathcal{C}^{\delta_1,0}([H], \{p\} \cup \underline{x}_V, J)$, then both C_1 and C_2 pass through $p \cup x_V$ which contains at least two points. Therefore, by the positivity of intersections, $C_1 \cdot C_2 = 2$. This is impossible because $C_1 \cdot C_2 = [H] \cdot [H] = 1$.

This contradiction implies that $\mathcal{C}^{\delta_1,0}(H, \{p\} \cup \underline{x}_V, J)$ also consists of a unique element. Thanks to the adjunction formula, this J -holomorphic curve is an embedding curve. \square

Let $\tilde{\mathbb{C}P}^2$ be the blow-up of $\mathbb{C}P^2$ at a point, and E the exceptional divisor. It is easy to see $\tilde{\mathbb{C}P}^2 \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$. Let $E_0 := \mathbb{P}_E(0 \oplus \mathcal{O})$, $E_\infty := \mathbb{P}_E(\mathcal{O}(-1) \oplus 0)$. E_0 and E_∞ are two distinguished non-intersecting sections of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$. One computes easily that

$$[E_\infty]^2 = -[E_0]^2 = 1.$$

The group $H_2(\tilde{\mathbb{C}P}^2, \mathbb{Z})$ is the free abelian group generated by $[E_\infty]$ and $[F]$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$. The first Chern class of $\tilde{\mathbb{C}P}^2$ is given by

$$c_1(\tilde{\mathbb{C}P}^2) = 3[E_\infty] - [E_0] = 2[E_\infty] + [F].$$

In $X = \tilde{\mathbb{C}P}^2 \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$, $V = E_\infty$, and $|\underline{x}^0| = 0$, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is always finite and made of simple maps.

Lemma 2.8. *Suppose that $X = \tilde{\mathbb{C}P}^2$ and $V = E_\infty$. Then the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ with $|\underline{x}^0| = 0$ is empty for a generic choice of J , except $\mathcal{C}^{\delta_1,0}([F], \underline{x}_V, J)$ which contains an unique element. Moreover, the unique element is an embedding.*

Proof. Suppose $d = a[E_\infty] + b[F]$. Since $c_1(\tilde{\mathbb{C}P}^2) = 2[E_\infty] + [F]$, we get

$$c_1(X) \cdot d - 1 - d \cdot [E_\infty] + |\beta| = 2a + b - 1 + |\beta|.$$

Suppose that $2a + b - 1 + |\beta| = 0$ and $\mathcal{C}^{\alpha,\beta}(a[E_\infty] + b[F], \underline{x}_V, J) \neq \emptyset$. Since $|\beta| \leq a + b$ and $|\beta| \geq 0$, we have $a + b \geq 0$. By the positivity of intersection,

we obtain

$$d \cdot [E_0] = (a[E_\infty] + b[F]) \cdot ([E_0]) = b \geq 0$$

$$d \cdot [F] = (a[E_\infty] + b[F]) \cdot [F] = a \geq 0.$$

We can deduce that $a = |\beta| = 0$, $b = |\alpha| = 1$.

The proof of the reminder of this lemma is similar to that in the proof of Lemma 2.7. □

3. BLOW-UP FORMULA OF WELSCHINGER INVARIANTS

3.1. Blow-up formula at a real point. In this subsection, we consider the behavior of Welschinger invariants under the blow-up of symplectic 4-manifold at a real point.

Let X be a compact real symplectic 4-manifold. Perform a real symplectic cut on X at the real point $x \in \mathbb{R}X$. We can get two real symplectic 4-manifolds $\bar{X}^+ \cong \mathbb{P}^2$ and $\bar{X}^- \cong X_{1,0}$ which contain a common real symplectic submanifold V . In \bar{X}^+ , $V \cong H$ is the hyperplane in \mathbb{P}^2 . In \bar{X}^- , $V \cong E$ is the exceptional divisor in $X_{1,0}$.

Let $\pi : \mathcal{Z} \rightarrow \Delta$ be the real symplectic sum of \bar{X}^+ and \bar{X}^- along V , $d \in H_2(\mathcal{Z}_\lambda; \mathbb{Z})$. Choose $\underline{x}(\lambda)$ a set of $c_1(X) \cdot d - 1$ real symplectic sections $\Delta \rightarrow \mathcal{Z}$ such that $\underline{x}(0) \cap V = \emptyset$. Choose an almost complex structure J on \mathcal{Z} tamed by $\omega_{\mathcal{Z}}$, which restrict to an almost structure J_λ tamed by ω_λ on each fiber \mathcal{Z}_λ , and generic with respect to all choices we made.

Let $X_\# = \bar{X}^+ \cup_V \bar{X}^-$. Denote $\mathcal{C}(d, \underline{x}(0), J_0)$ to be the set $\{\bar{f} : \bar{C} \rightarrow X_\#\}$ of limits, as stable maps, of maps in $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0, where $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ is the set of all irreducible rational J -holomorphic curves in $(\mathcal{Z}_\lambda, \omega_\lambda, J_\lambda)$ passing through all points in $\underline{x}(\lambda)$, realizing the class d . From [14, Section 3], we know \bar{C} is a connected nodal rational curve such that:

- $\underline{x}(0) \subset \bar{f}(\bar{C})$;
- any point $p \in \bar{f}^{-1}(V)$ is a node of \bar{C} which is the intersection of two irreducible components \bar{C}' and \bar{C}'' of \bar{C} , with $\bar{f}(\bar{C}') \subset X^+$ and $\bar{f}(\bar{C}'') \subset X^-$;
- if in addition neither $\bar{f}(\bar{C}')$ nor $\bar{f}(\bar{C}'')$ is entirely mapped into V , then the multiplicities of intersection of both $\bar{f}(\bar{C}')$ and $\bar{f}(\bar{C}'')$ with V are equal.

Given an element $\bar{f} : \bar{C} \rightarrow X_\#$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, denote by C_* , $* = +, -$, the union of the irreducible components of \bar{C} mapped into \bar{X}^* .

Proposition 3.1. *Assume $\underline{x}(0) \cap \bar{X}^+$ contains at most one point, $\underline{x}(0) \cap \bar{X}^- \neq \emptyset$ if $\underline{x}(0) \cap \bar{X}^+ \neq \emptyset$. Then for a generic J_0 , the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0 . Given an element $\bar{f} : \bar{C} \rightarrow X_\#$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover the following are true:*

- (1) If $\underline{x}(0) \cap \bar{X}^+ = \emptyset$, then C_+ is empty. The curve C_- is irreducible, and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.
- (2) If $\underline{x}(0) \cap \bar{X}^+ = \{p\}$, then C_+ is irreducible and $\bar{f}(C_+)$ realizes class $[H]$. The curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d - [E], \underline{x}(0) \cap \bar{X}^-, J_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.

Proof. From Example 11.4 and Lemma 14.6 of [14], we know that no component of \bar{C} is entirely mapped into V , also see [7].

Note that $[E]^2 = -1$ in the real blow-up $\bar{X}^- = X_{1,0}$, and $c_1(\bar{X}^-) \cdot [E] = 1$. Suppose $\bar{f}_*[C_+] = a[H]$, $a \geq 0$, $\bar{f}_*[C_-] = p^!d - b[E]$, $b \geq 0$. Then we have

$$a = \bar{f}_*[C_+] \cdot [H] = (p^!d - b[E]) \cdot [E] = b.$$

Since $\underline{x}(0) \cap \bar{X}^+$ contains at most one point, we will consider the two cases separately.

Case I: $\underline{x}(0) \cap \bar{X}^+ = \emptyset$.

In this case, we know that $\bar{f}(C_-)$ passes through all the $c_1(X) \cdot d - 1$ points in $\underline{x}(0) \cap \bar{X}^-$ and realizes the class $p^!d - b[E]$ in $H_2(\bar{X}^-; \mathbb{Z})$. Suppose C_- consists of irreducible components $\{C_{-i}\}_{i=1}^m$, and there are $0 \leq k \leq m$ irreducible components $\{C_{-i}\}_{i=1}^k$ such that the restriction $\bar{f}|_{C_{-i}}$, $i = 1, \dots, k$, is non-simple which factors through a non-trivial ramified covering of degree $\delta_i \geq 2$ of a simple map $f_i : \mathbb{P}^1 \rightarrow X^-$. Assume $(f_i)_*[\mathbb{P}^1] = d_i$, $i = 1, \dots, k$, and $\bar{f}_*[C_{-j}] = d_j$, $j = k+1, \dots, m$. then $\sum_{i=1}^k \delta_i d_i + \sum_{j=k+1}^m d_j = p^!d - b[E]$.

$$\begin{aligned} & c_1(\bar{X}^-) \cdot \left(\sum_{i=1}^k d_i \right) - k + c_1(\bar{X}^-) \cdot \left(\sum_{j=k+1}^m d_j \right) - (m - k) \\ & \geq c_1(X) \cdot d - 1 \\ & = c_1(\bar{X}^-) \cdot \left(\sum_{i=1}^k \delta_i d_i + \sum_{j=k+1}^m d_j \right) + b - 1. \end{aligned}$$

Therefore,

$$(12) \quad \sum_{i=1}^k (1 - \delta_i) c_1(\bar{X}^-) \cdot d_i \geq m + b - 1.$$

Since $c_1(\bar{X}^-) \cdot d_i \geq 0$, $b \geq 0$, $\delta_i \geq 2$, so (12) holds only when $m = 1, k = 0$ and $b = 0$. This implies that C_- is irreducible and $\bar{f}|_{C_-}$ is simple. $b = 0$ also implies $\bar{f}_*[C_+] = 0$. Therefore, $C_+ = \emptyset$.

The previous argument implies that $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)$. Moreover, the finiteness of $\mathcal{C}^{0,0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)$ implies that $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite.

Case II : $\underline{x}(0) \cap \bar{X}^+ = \{p\}$.

In this case, the fact that the image of $\bar{f}(C_+)$ has to pass $\{p\}$ implies $a = b \geq 1$. $\bar{f}(C_-)$ passes through all the $c_1(X) \cdot d - 2$ points in $\underline{x}(0) \cap X^-$ and realizes the class $p^!d - b[E]$ in $H_2(\bar{X}^-; \mathbb{Z})$. Similar to case I, we know that C_- is irreducible. Next, we prove that $\bar{f}|_{C_-}$ is simple. For this, we assume that $\bar{f}|_{C_-}$ is non-simple. Then $\bar{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{P}^1 \rightarrow \bar{X}^-$, and $(f_0)_*[\mathbb{P}^1] = \frac{1}{\delta}(p^!d - b[E])$. Therefore

$$\frac{1}{\delta}c_1(\bar{X}^-) \cdot (p^!d - b[E]) - 1 \geq c_1(X) \cdot d - 2.$$

So we have

$$(13) \quad c_1(X) \cdot d + \delta - \delta c_1(X) \cdot d \geq b.$$

Since $\delta \geq 2$, $c_1(X) \cdot d \geq 2$, (13) implies $b \leq 0$. This is in contradiction with $b \geq 1$. Therefore, $\bar{f}|_{C_-}$ is simple.

From

$$\begin{aligned} c_1(\bar{X}^-) \cdot (p^!d - b[E]) - 1 &= c_1(X) \cdot d - 1 - b \\ &\geq c_1(X) \cdot d - 2. \end{aligned}$$

we obtain $b \leq 1$. So we have $b = 1$, and $\bar{f}_*[C_+] = [H]$.

The previous argument implies that $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, \delta_1}(p^!d - [E], \underline{x}(0) \cap \bar{X}^-, J_0)$. The finiteness of $\mathcal{C}^{0, \delta_1}(p^!d - [E], \underline{x}(0) \cap \bar{X}^-, J_0)$ implies that $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite.

The number of elements of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to \bar{f} as λ goes to 0 follows from [14]. Let's review the behavior of an elements $f_\lambda : C_\lambda \rightarrow \mathcal{Z}_\lambda$ of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to \bar{f} close to the smoothing of the intersection point p of C_- and C_+ . In local coordinates (λ, x, y) at $\bar{f}(p)$, the manifold \mathcal{Z} is given by the equation $xy = \lambda$. Locally,

$$\bar{X}^+ = \{\lambda = 0 \text{ and } y = 0\}, \quad \bar{X}^- = \{\lambda = 0 \text{ and } x = 0\}.$$

Since the order of intersection of \bar{f}_{C_+} and V at $\bar{f}(p)$ is 1, the maps \bar{f}_{C_+} and \bar{f}_{C_-} have expansions

$$x(z) = mz + o(z) \quad \text{and} \quad y(w) = nw + o(w),$$

where z and w are local coordinates at p of C_+ and C_- respectively.

For $0 < |\lambda| \ll 1$, there exists a solution $\mu(\lambda) \in \mathbb{C}^*$ of

$$\mu(\lambda) = \frac{\lambda}{mn}$$

such that the smoothing of \bar{C} at p is locally given by $zw = \mu(\lambda)$, and the map f_λ is approximated by the map

$$\{zw = \mu(\lambda)\} \subset \mathbb{C}^2 \mapsto (\lambda, mz, nw)$$

close to the smoothing of p . Furthermore, such maps $f_\lambda \in \mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to \bar{f} are in one to one correspondence with the choice of such $\mu(\lambda)$ for each point of $C_+ \cap C_-$.

□

Applying Proposition 3.1, we can get a comparison theorem of the Welschinger invariants. Let $\mathbb{R}\mathcal{C}^{\alpha^r+\alpha^c, \beta^r+\beta^c}(d, \underline{x}, J)$ be the set of real rational curves in $\mathcal{C}^{\alpha, \beta}(d, \underline{x}, J)$, $\alpha = \alpha^r + \alpha^c$ and $\beta = \beta^r + \beta^c$, such that the α (or β) "points" consists of α^r (or β^r) real "points" and $\frac{1}{2}\alpha^c$ (or $\frac{1}{2}\beta^c$) pairs of τ -conjugated "points".

Proposition 3.2. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \rightarrow X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Let $\underline{x}(\lambda)$, \bar{X}^+ , \bar{X}^- , J_0 be as before. Then*

$$(14) \quad \bullet \text{ if } \underline{x}(0) \cap \bar{X}^+ = \{p\}, \underline{x}(0) \cap \bar{X}^- \neq \emptyset, \\ W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0, \delta_1^r}(p^!d - [E], \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_-)},$$

$$(15) \quad \bullet \text{ if } \underline{x}(0) \cap \bar{X}^+ = \emptyset, \\ W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_-)}.$$

where E is the exceptional divisor.

Proof. Equip the small disc Δ with the standard complex conjugation. From subsection 2.3, we know one can equip the symplectic sum $\pi : \mathcal{Z} \rightarrow \Delta$ with a real structure $\tau_{\mathcal{Z}}$ which is induced by the real structures τ_- , τ_+ on the real symplectic cuts \bar{X}^- and \bar{X}^+ such that the map $\pi : \mathcal{Z} \rightarrow \Delta$ is real. Choose a set of real sections $\underline{x} : \Delta \rightarrow \mathcal{Z}$. Let $\bar{f} : \bar{C} \rightarrow X_{\sharp}$ be a real element in $\mathbb{R}\mathcal{C}(d, \underline{x}(0), J_0)$.

For the case $\underline{x}(0) \cap \bar{X}^+ = \{p\}$ and $\underline{x}(0) \cap \bar{X}^- \neq \emptyset$, from Proposition 3.1 and Lemma 2.7 we know $\bar{f}_*[C_+] = [H]$ and $\bar{f}|_{C_+}$ is an embedded simple curve. $\bar{f}(C_+)$ has no self-intersection point, so $\bar{f}(C_+)$ has no node. And there is only one possibility for $\bar{f}|_{C_+}$ to recover a real curve $\bar{f}(\bar{C})$ when $\bar{f}|_{C_-}$ is fixed. In other words, the number of real curves $\bar{f} \in \mathbb{R}\mathcal{C}(d, \underline{x}(0), J_0)$ is equal to the number of the real curves $\bar{f}|_{C_-} \in \mathbb{R}\mathcal{C}^{0, \delta_1}(p^!d - [E], \underline{x}(0) \cap \bar{X}^-, J_0)$. Therefore we have

$$(16) \quad \begin{aligned} m_{X_{\sharp}}(\bar{f}(\bar{C})) &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}) + m_{\bar{X}^+}(\bar{f}|_{C_+}) \\ &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}). \end{aligned}$$

By Proposition 3.1, an element \bar{f} of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_{\lambda})$, so the latter has to be real when \bar{f} is real and $\lambda \in \mathbb{C}^*$ is small. The description at the end of the proof of Proposition 3.1 of the local deformation of \bar{f} shows that no node appears in a neighborhood of $V \cap \bar{f}(\bar{C})$ when deforming \bar{f} . Combining with (16), this implies (14).

For the case $\underline{x}(0) \cap \bar{X}^+ = \emptyset$, we know $\bar{f}_*[C_+] = 0$ from Proposition 3.1. The real curve $\bar{f}(\bar{C})$ is determined by the part $\bar{f}|_{C_-}$. In other words, the number of real curves $\bar{f} \in \mathbb{R}\mathcal{C}(d, \underline{x}(0), J_0)$ is equal to the number of the real

curves $\bar{f}|_{C_-} \in \mathbb{R}\mathcal{C}^{0,0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)$. Then we have

$$m_{X_{\sharp}}(\bar{f}(\bar{C})) = m_{X^-}(\bar{f}|_{\bar{C}^-})$$

The rest of the Proposition can be proved similar to the previous case. \square

Remark 3.3. Proposition 3.2 tells us that the sum

$$\sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^T}(p^!d-[E], \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_-)}$$

on the right side of formula (14) does not dependent on $\underline{x}(0) \cap \bar{X}^-$ and J_0 . It can be seen as a particular case of relative Welschinger invariants. See [3] and [22] for more about relative Welschinger invariants.

Now we take real symplectic cut along the exceptional divisor E of the real symplectic blow-up manifold $\tilde{X} = X_{1,0}$. We can get two real symplectic cuts:

$$\bar{X}_{1,0}^+ \cong \mathbb{P}(N_{E|X_{1,0}} \oplus \mathcal{O}_E) \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \quad \text{and} \quad \bar{X}_{1,0}^- \cong X_{1,0},$$

which contain a common real symplectic submanifold V . In $\bar{X}_{1,0}^+$, $V \cong E_{\infty}$ is the infinity section of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$. In $\bar{X}_{1,0}^- \cong X_{1,0}$, $V \cong E$ is the exceptional divisor.

Let \tilde{Z} be the real symplectic sum of $\bar{X}_{1,0}^+$ and $\bar{X}_{1,0}^-$ along V . Let $p^!d - [E] \in H_2(\tilde{Z}_{\lambda}; \mathbb{Z})$, where $d \in H_2(X; \mathbb{Z})$. Choose $\tilde{x}_1(\lambda)$ a set of $c_1(X) \cdot d - 1$ real symplectic sections $\Delta \rightarrow \tilde{Z}$ such that $\tilde{x}_1(0) \cap V = \emptyset$, $\tilde{x}_1(0) \cap \bar{X}_{1,0}^+ = \emptyset$. Choose $\tilde{x}_2(\lambda)$ a set of $c_1(X) \cdot d - 2$ real symplectic sections $\Delta \rightarrow \tilde{Z}$ such that $\tilde{x}_2(0) \cap V = \emptyset$, $\tilde{x}_2(0) \cap \bar{X}_{1,0}^+ = \emptyset$. Choose a generic almost complex structure \tilde{J} on \tilde{Z} as above.

Let $\tilde{X}_{\sharp} = \bar{X}_{1,0}^+ \cup_V \bar{X}_{1,0}^-$. Define $\mathcal{C}(p^!d, \tilde{x}_1(0), \tilde{J}_0)$, $\mathcal{C}(p^!d - [E], \tilde{x}_2(0), \tilde{J}_0)$ to be the set $\{\bar{f} : \bar{C} \rightarrow \tilde{X}_{\sharp}\}$ of limits, as stable maps, of maps in $\mathcal{C}(p^!d, \tilde{x}_1(\lambda), \tilde{J}_{\lambda})$, $\mathcal{C}(p^!d - [E], \tilde{x}_2(\lambda), \tilde{J}_{\lambda})$ as λ goes to 0, respectively. Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_{\sharp}$ of $\mathcal{C}(p^!d, \tilde{x}_2(0), \tilde{J}_0)$ or $\mathcal{C}(p^!d - [E], \tilde{x}_2(0), \tilde{J}_0)$, Denote by C_* , $* = +, -$, the union of the irreducible components of \bar{C} mapped to $\bar{X}_{1,0}^*$.

Proposition 3.4. *Under the assumption above, we have the following:*

(1) *For a generic \tilde{J}_0 , the set $\mathcal{C}(p^!d, \tilde{x}_1(0), \tilde{J}_0)$ is finite, and only depends on $\tilde{x}_1(0)$ and \tilde{J}_0 . Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_{\sharp}$ of $\mathcal{C}(p^!d, \tilde{x}_1(0), \tilde{J}_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d, \tilde{x}_1(0) \cap \bar{X}_{1,0}^-, \tilde{J}_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(p^!d, \tilde{x}_1(\lambda), \tilde{J}_{\lambda})$ as λ goes to 0.*

(2) *For a generic \tilde{J}_0 , the set $\mathcal{C}(p^!d - [E], \tilde{x}_2(0), \tilde{J}_0)$ is finite, and only depends on $\tilde{x}_2(0)$ and \tilde{J}_0 . Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_{\sharp}$ of $\mathcal{C}(p^!d - [E], \tilde{x}_2(0), \tilde{J}_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map,*

and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d - [E], \tilde{\mathcal{X}}_2(0) \cap \bar{X}_{1,0}^-, \tilde{J}_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(p^!d - [E], \tilde{\mathcal{X}}_2(\lambda), \tilde{J}_\lambda)$ as λ goes to 0.

Proof. The fact that no component of \bar{C} is entirely mapped into V follows from [14, Example 11.4 and Lemma 14.6], also see [7].

(1) Suppose $\bar{f}_*[C_+] = a[F] + b[E_\infty]$, $\bar{f}_*[C_-] = p^!d - k[E]$, $k \geq 0$ where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$ with $F \cdot [E_0] = 1$ and $F \cdot [E_\infty] = 1$. Then

$$\begin{aligned} a + b &= (a[F] + b[E_\infty]) \cdot [E_\infty] = (p^!d - k[E]) \cdot [E] = k, \\ a &= (a[F] + b[E_\infty]) \cdot [E_0] = (p^!d) \cdot [E] = 0. \end{aligned}$$

In $\bar{X}_{1,0}^-$, we know $\bar{f}|_{C_-}$ passes through

$$|\tilde{\mathcal{X}}_1(0)| = c_1(X) \cdot d - 1 = c_1(\bar{X}_{1,0}^-) \cdot (p^!d) - 1$$

distinct points in $\bar{X}_{1,0}^-$. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

Next, we prove that $\bar{f}|_{C_-}$ is simple. For this, we assume that $\bar{f}|_{C_-}$ is non-simple. Then $\bar{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{P}^1 \rightarrow \bar{X}_{1,0}^-$, and $(f_0)_*[\mathbb{P}^1] = \frac{1}{\delta}(p^!d - k[E])$. Therefore,

$$\begin{aligned} \frac{1}{\delta} c_1(\bar{X}_{1,0}^-) \cdot (p^!d - k[E]) - 1 &\geq c_1(\bar{X}_{1,0}^-) \cdot (p^!d) - 1 \\ (17) \quad (1 - \delta) c_1(\bar{X}_{1,0}^-) \cdot (p^!d) &\geq k. \end{aligned}$$

Since $c_1(X) \cdot d \geq 1$, $\delta \geq 2$, $k \geq 0$, so (17) is impossible. Therefore, $\bar{f}|_{C_-}$ can only be simple.

On the other hand, we have

$$\begin{aligned} c_1(\bar{X}_{1,0}^-) \cdot (p^!d - k[E]) - 1 &= c_1(\bar{X}_{1,0}^-) \cdot p^!d - k - 1 \\ &\geq c_1(\bar{X}_{1,0}^-) \cdot (p^!d) - 1. \end{aligned}$$

This implies $k = 0$ and $b = 0$. Therefore, $C_+ = \emptyset$ and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d, \tilde{\mathcal{X}}_1(0) \cap \bar{X}_{1,0}^-, \tilde{J}_0)$ which is also a simple map.

(2) Suppose $\bar{f}_*[C_+] = a[F] + b[E_\infty]$, $\bar{f}_*[C_-] = p^!d - k[E]$, $k \geq 1$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$ with $F \cdot [E_0] = 1$ and $F \cdot [E_\infty] = 1$. Then

$$\begin{aligned} a + b &= (a[F] + b[E_\infty]) \cdot [E_\infty] = (p^!d - k[E]) \cdot [E] = k, \\ a &= (a[F] + b[E_\infty]) \cdot [E_0] = (p^!d - [E]) \cdot [E] = 1. \end{aligned}$$

In $\bar{X}_{1,0}^-$, we know that $\bar{f}|_{C_-}$ passes through

$$c_1(X) \cdot d - 2 = c_1(\bar{X}_{1,0}^-) \cdot (p^!d - [E]) - 1$$

distinct points. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

By a similar analysis of the dimension condition as in the proof of case (1), we obtain $\bar{f}_*[C_+] = [F]$, C_+ must have exactly 1 component and the image of it realize the class $[F]$. $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d - [E], \tilde{x}_2(0) \cap \overline{X}_{1,0}^-, \tilde{J}_0)$ which is a simple map.

The proof of other parts is the same as that in the proof of Proposition 3.1. We omit it here. \square

Proposition 3.5. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d - 1 = r + 2s > 0$ and $\tau_*d = -d$. Denote by $p : X_{1,0} \rightarrow X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Let $\tilde{x}_1, \tilde{x}_2, \tilde{J}_0, \overline{X}_{1,0}^+, \overline{X}_{1,0}^-$ be as before. Then*

$$\begin{aligned} W_{X_{1,0}}(p^!(d), s) &= \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0}(p^!d, \tilde{x}_1(0) \cap \overline{X}_{1,0}^-, \tilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)}, \\ W_{X_{1,0}}(p^!(d) - [E], s) &= \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^+}(p^!d - [E], \tilde{x}_2(0) \cap \overline{X}_{1,0}^-, \tilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)}, \end{aligned}$$

where E is the exceptional divisor.

Remark 3.6. Similar to Proposition 3.2, by Proposition 3.4, one can prove Proposition 3.5. Moreover, Proposition 3.2 and Proposition 3.5 imply Theorem 1.1.

3.2. Blow-up formula at a conjugated pair. Let (X, ω) be a compact connected real symplectic 4-manifold, and let $V_1, V_2 \subset X$ be two disjoint embedded symplectic curves in X . Let $d \in H_2(X; \mathbb{Z})$ and $\alpha^1, \alpha^2, \beta^1, \beta^2 \in \mathbb{Z}_{\geq 0}^\infty$ such that

$$I\alpha^1 + I\beta^1 = d \cdot [V_1], \quad I\alpha^2 + I\beta^2 = d \cdot [V_2].$$

Choose a configuration $\underline{x} = \underline{x}^\circ \sqcup \underline{x}_{V_1} \sqcup \underline{x}_{V_2}$ of points in X , with \underline{x}° a configuration of $c_1(X) \cdot d - 1 - d \cdot ([V_1] + [V_2]) + |\beta_1| + |\beta_2|$ points in $X \setminus (V_1 \cup V_2)$, $\underline{x}_{V_1} = \{p_{i,j}\}_{0 < j \leq \alpha_i^1, i \geq 1}$ a configuration of $|\alpha^1|$ points in V_1 , and $\underline{x}_{V_2} = \{q_{i,j}\}_{0 < j \leq \alpha_i^2, i \geq 1}$ a configuration of $|\alpha^2|$ points in V_2 . Given an ω -tamed almost complex structure J on X such that V_1 and V_2 are J -holomorphic, denote by $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ the set of rational J -holomorphic maps $f : \mathbb{C}P^1 \rightarrow X$ such that

- $f_*[\mathbb{C}P^1] = d$;
- $\underline{x} \subset f(\mathbb{C}P^1)$;
- $V_1 \cup V_2$ does not contain $f(\mathbb{C}P^1)$;
- $f(\mathbb{C}P^1)$ has order of contact i with V_1 at each points $p_{i,j}$ and has order of contact i with V_2 at each points $q_{i,j}$;
- $f(\mathbb{C}P^1)$ has order of contact i with V_1 at exactly β_i^1 distinct points on $V_1 \setminus \underline{x}_{V_1}$ and has order of contact i with V_2 at exactly β_i^2 distinct points on $V_2 \setminus \underline{x}_{V_2}$.

Note that $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ may contain components of positive dimension corresponding to non-simple maps. But for the generic J , the set of simple maps in $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ is 0-dimensional.

Lemma 3.7. *Let (X, ω) be a compact connected real symplectic 4-manifold. Suppose that V_1 and V_2 are two embedded real symplectic spheres in X with $V_1 \cdot V_2 = 0$, $[V_i]^2 = -1$, $i = 1, 2$, and assume $|\beta^i| = d \cdot [V_i]$, $i = 1, 2$. Then for a generic choice of J , the set $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ is finite and contains only simple maps that are all immersions.*

Proof. Suppose that $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ contains a non-simple map which factors through a non-trivial ramified covering of degree δ of a simple map $f_0 : \mathbb{C}P^1 \rightarrow X$. Let d_0 denotes the homology class $(f_0)_*[\mathbb{C}P^1]$. Since $f_0(\mathbb{C}P^1)$ passes through $c_1(X) \cdot d - 1 = \delta c_1(X) \cdot d_0 - 1$ points, we have

$$c_1(X) \cdot d_0 - 1 \geq \delta c_1(X) \cdot d_0 - 1 \geq 0,$$

which is impossible.

Suppose that $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ contains infinitely many simple maps. By Gromov compactness Theorem, There exists a sequence $(f_n)_{n \geq 0}$ of simple maps in $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ which converges to some J -holomorphic map $\bar{f} : \bar{C} \rightarrow X$. By genericity of J , the set of simple maps in $\mathcal{C}^{\alpha^1, \beta^1, \alpha^2, \beta^2}(d, \underline{x}, J)$ is discrete. Hence either \bar{C} is reducible, or \bar{f} is non-simple. Let $\bar{C}_1, \dots, \bar{C}_m, \bar{C}_1^1, \dots, \bar{C}_{m_1}^1, \bar{C}_1^2, \dots, \bar{C}_{m_2}^2$ be the irreducible components of \bar{C} , labeled in such a way that

- $\bar{f}(\bar{C}_i) \not\subseteq V_1 \cup V_2$;
- $\bar{f}(\bar{C}_i^j) \subset V_j$ and $(\bar{f})_*[\bar{C}_i^j] = k_i^j[V_j]$, $j = 1, 2$.

Define $k^j = \sum_{i=1}^{m_j} k_i^j$, $j = 1, 2$. The restriction of \bar{f} to $\cup_{i=1}^m \bar{C}_i$ is subject to $c_1(X) \cdot d - 1 - d \cdot ([V_1] + [V_2]) + |\beta^1| + |\beta^2|$ points constrains, so we have

$$c_1(X) \cdot (d - k^1[V_1] - k^2[V_2]) - m \geq c_1(X) \cdot d - 1 - d \cdot ([V_1] + [V_2]) + |\beta^1| + |\beta^2|.$$

Since both V_1 and V_2 are embedded real symplectic spheres, from adjunction formula, we can get $c_1(X) \cdot [V_i] = 1$, $i = 1, 2$. Hence we obtain

$$c_1(X) \cdot d - k^1 - k^2 - m \geq c_1(X) \cdot d - 1 - d \cdot ([V_1] + [V_2]) + |\beta^1| + |\beta^2|.$$

From the assumption of Lemma, we have

$$0 \geq m + k^1 + k^2 - 1.$$

Therefore, we have $m = 1$, $k^1 = k^2 = 0$. The map \bar{f} has to be a non-simple map which factors through a non-trivial ramified covering of a simple map $f_0 : \mathbb{C}P^1 \rightarrow X$. But f_0 is subject to more point constraints, which provides a contradiction. \square

Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, and suppose $y_1, y_2 \in X \setminus \mathbb{R}X$ is a τ -conjugated pair. Denote by $p : X_{0,1} \rightarrow X$ the projection of

the real symplectic blow-up of X at y_1, y_2 . Perform real symplectic cut of X at the τ -conjugated pair y_1, y_2 . We get

$$\bar{X}^+ = \bar{X}^{+1} \sqcup \bar{X}^{+2} \cong \mathbb{P}^2 \sqcup \mathbb{P}^2, \quad \bar{X}^- \cong X_{0,1}.$$

Both \bar{X}^+ and \bar{X}^- contain two common real symplectic submanifolds V_1, V_2 of real codimension 2. Let $V = V_1 \sqcup V_2$. In \bar{X}^+ , $V_1 \cong H_1, V_2 \cong H_2$, where H_i is the hyperplane of $\bar{X}^{+i}, i = 1, 2$. $V_1 \cong E_1$ and $V_2 \cong E_2$ are the associated exceptional divisors in \bar{X}^- at $y_i, i = 1, 2$, respectively.

Let \mathcal{Z} be the real symplectic sum of the two real symplectic manifolds \bar{X}^+ and \bar{X}^- along V , and $d \in H_2(\mathcal{Z}; \mathbb{Z})$. Denote $\underline{x}(\lambda), J, X_\# = \bar{X}^+ \cup_V \bar{X}^-, \mathcal{C}(d, \underline{x}(0), J_0), C_{+i}, i = 1, 2$, as in subsection 3.1. Similar to the proof of Proposition 3.1, we can prove

Proposition 3.8. *Assume $\underline{x}(0) \cap \bar{X}^{+i}$ contains at most one point, $\underline{x}(0) \cap \bar{X}^- \neq \emptyset$ if $\underline{x}(0) \cap \bar{X}^+ \neq \emptyset$. Let $\underline{x}(0), d, J$ be given above. Then for a generic J_0 , the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0 . Given an element $\bar{f} : \bar{C} \rightarrow X_\#$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the followings are true:*

- (1) *If $\underline{x}(0) \cap \bar{X}^{+i} = \{p_i\}, i = 1, 2$, the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, \delta_1, 0, \delta_1}(p^!d - [E_1] - [E_2], \underline{x}(0) \cap \bar{X}^-, J_0)$. The curves $C_{+i}, i = 1, 2$, are irreducible and the image of C_{+i} represses $[H_i]$ and passes $\{p_i\}$, respectively. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.*
- (2) *If $\underline{x}(0) \cap \bar{X}^{+i} = \emptyset, i = 1, 2$, then $C_{+i} = \emptyset$, the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 0, 0, 0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.*

Base on Proposition 3.8, similar to Proposition 3.2, we can prove

Proposition 3.9. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_*d = -d$. Suppose $y_1, y_2 \in (X \setminus \mathbb{R}X)$ is a τ -conjugated pair. Denote by $p : X_{0,1} \rightarrow X$ the projection of the real symplectic blow-up of X at y_1, y_2 . Then*

- (1) *If $\underline{x}(0) \cap \bar{X}^{+i} = \emptyset, i = 1, 2$, then*

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0, 0, 0, 0}(p^!d, \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{0,1}}(C_-)}.$$

- (2) *If $\underline{x}(0) \cap \bar{X}^{+i} = \{p_i\}, i = 1, 2, \underline{x}(0) \cap \bar{X}^- \neq \emptyset$, then*

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0, \delta_1^c, 0, \delta_1^c}(p^!d - [E_1] - [E_2], \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{0,1}}(C_-)},$$

where E_1, E_2 are the exceptional divisors.

Next, perform the real symplectic cut of $X_{0,1}$ along E_1, E_2 , where E_i is the exceptional divisor of the blow-up at $y_i, i = 1, 2$, respectively. We obtain

two real symplectic manifolds $\overline{X}^+ = \overline{X}^{+1} \sqcup \overline{X}^{+2}$ and \overline{X}^- as follow:

$$\overline{X}^+ \cong \mathbb{P}_{E_1}(\mathcal{O}(-1) \oplus \mathcal{O}) \sqcup \mathbb{P}_{E_2}(\mathcal{O}(-1) \oplus \mathcal{O}), \quad \overline{X}^- \cong X_{0,1}.$$

Both \overline{X}^+ and \overline{X}^- contain a common real symplectic submanifold V_1, V_2 of real codimension 2 respectively. Let $V = V_1 \sqcup V_2$. In \overline{X}^+ , $V_1 \cong E_\infty^1$, $V_2 \cong E_\infty^2$, where E_∞^i is the infinity section of $\mathbb{P}_{E_i}(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E_i$, respectively. $V_1 \cong E_1$, $V_2 \cong E_2$ are the exceptional divisors in \overline{X}^- .

Let $\tilde{\mathcal{Z}}$ be the symplectic sum of the two real symplectic manifolds \overline{X}^+ and \overline{X}^- along V . Let $p^!d - [E_1] - [E_2] \in H_2(\tilde{\mathcal{Z}}_\lambda; \mathbb{Z})$, where $d \in H_2(X; \mathbb{Z})$. Choose $\tilde{x}_1(\lambda)$ a set of $c_1(X) \cdot d - 3$ real symplectic sections $\Delta \rightarrow \tilde{\mathcal{Z}}$ such that $\tilde{x}_1(0) \cap V = \emptyset$, $\tilde{x}_1(0) \cap \overline{X}^+ = \emptyset$. Choose $\tilde{x}_2(\lambda)$ a set of $c_1(X) \cdot d - 1$ real symplectic sections $\Delta \rightarrow \tilde{\mathcal{Z}}$ such that $\tilde{x}_2(0) \cap V = \emptyset$, $\tilde{x}_2(0) \cap \overline{X}^+ = \emptyset$. Choose a generic almost complex structure \tilde{J} on $\tilde{\mathcal{Z}}$ as above.

Denote $\tilde{X}_\# = \overline{X}^+ \cup_V \overline{X}^-$, $\mathcal{C}(p^!d - [E_1] - [E_2], \tilde{x}_1(0), \tilde{J}_0)$, $\mathcal{C}(p^!d, \tilde{x}_2(0), \tilde{J}_0)$, C_{+i} , $i = 1, 2$ and C_- as in subsection 3.1. The same argument as in the proof of Proposition 3.1 and Proposition 3.2 shows that Proposition 3.10 and Proposition 3.11 hold.

Proposition 3.10. *Let $\tilde{x}(0)$, $p^!d - [E_1] - [E_2]$, \tilde{J} be given above. Then we have*

(1) *For a generic \tilde{J}_0 , the set $\mathcal{C}(p^!d - [E_1] - [E_2], \tilde{x}_1(0), \tilde{J}_0)$ is finite, and only depends on $\tilde{x}_1(0)$ and \tilde{J}_0 . Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_\#$ of $\mathcal{C}(p^!d - [E_1] - [E_2], \tilde{x}_1(0), \tilde{J}_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the curve $\bar{f}|_{C_-}$ is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, \delta_1, 0, \delta_1}(p^!d - [E_1] - [E_2], \tilde{x}_1(0) \cap \overline{X}^-, \tilde{J}_0)$. C_{+i} , $i = 1, 2$, are irreducible and the image of C_{+i} under \bar{f} represents the fiber class $[F_i]$ of $\mathbb{P}_{E_i}(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E_i$, respectively. The map \bar{f} is the limit of a unique element of $\mathcal{C}(p^!d - [E_1] - [E_2], \tilde{x}_1(\lambda), \tilde{J}_\lambda)$ as λ goes to 0.*

(2) *For a generic \tilde{J}_0 , the set $\mathcal{C}(p^!d, \tilde{x}_2(0), \tilde{J}_0)$ is finite, and only depends on $\tilde{x}_2(0)$ and \tilde{J}_0 . Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_\#$ of $\mathcal{C}(p^!d, \tilde{x}_2(0), \tilde{J}_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the curve $\bar{f}|_{C_-}$ is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 0, 0, 0}(p^!d, \tilde{x}_2(0) \cap \overline{X}^-, \tilde{J}_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(p^!d, \tilde{x}_2(\lambda), \tilde{J}_\lambda)$ as λ goes to 0.*

Proposition 3.11. *Let $X_\mathbb{R}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_*d = -d$. Suppose $y_1, y_2 \in (X \setminus \mathbb{R}X)$ is a τ -conjugated pair. Denote by $p : X_{0,1} \rightarrow X$ the projection of the real*

symplectic blow-up of X at y_1, y_2 . Then

$$W_{X_{0,1}}(p^!d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0,0,0}(p^!d, \tilde{x}_2(0) \cap \tilde{X}^-, \tilde{J}_0)} (-1)^{m_{X_{0,1}}(C_-)}.$$

Moreover, if $s \geq 1$, then

$$\begin{aligned} & W_{X_{0,1}}(p^!d - [E_1] - [E_2], s - 1) \\ &= \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^c,0,\delta_1^c}(p^!d - [E_1] - [E_2], \tilde{x}_1(0) \cap \tilde{X}^-, \tilde{J}_0)} (-1)^{m_{X_{0,1}}(C_-)}, \end{aligned}$$

where E_1, E_2 are the exceptional divisors.

Remark 3.12. Proposition 3.9 and Proposition 3.11 implies Theorem 1.2.

4. WALL-CROSSING FORMULA OF WELSCHINGER INVARIANTS

4.1. Wall-crossing formula. Welschinger [36] introduced a new invariant $\theta_{X_{\mathbb{R}}}(d, s)$ to describe the variation of Welschinger invariants when replacing a pair of real fixed points in the same component of $\mathbb{R}X$ by a pair of τ -conjugated points. Welschinger proved the following wall-crossing formula, Theorem 3.2 in [36],

Theorem 4.1. ([36]) *Let (X, ω, τ) be a compact real symplectic 4-manifold such that $\mathbb{R}X$ is connected, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d - 1 > 0$ and $\tau_*d = -d$, and s be an integer between 1 and $\lfloor \frac{c_1(X) \cdot d - 1}{2} \rfloor$. Then*

$$W_{X_{\mathbb{R}}}(d, s - 1) = W_{X_{\mathbb{R}}}(d, s) + 2\theta_{X_{\mathbb{R}}}(d, s - 1).$$

In algebraic geometry category, Itenberg, Kharlamov and Shustin [16] observed that The invariant $\theta_{X_{\mathbb{R}}}(d, s)$ can be considered as the Welschinger invariants on the blow-up at the fixed real point. In the following, we will use the degeneration technique to verify this observation for any symplectic 4-manifolds.

Perform the real symplectic cut on X at the real point $x \in \mathbb{R}X$. We can get

$$\bar{X}^+ \cong \mathbb{P}^2 \quad \text{and} \quad \bar{X}^- \cong X_{1,0}.$$

In this section, we assume that $d \in H_2(X, \mathbb{Z})$ such that $c_1(X) \cdot d \geq 4$ and $\tau_*d = -d$. Denote $\pi : \mathcal{Z} \rightarrow \Delta$, $\underline{x}(\lambda)$, J , X_{\sharp} , $\mathcal{C}(d, \underline{x}(0), J_0)$, C_* , $*$ = +, -, as in subsection 3.1. First of all, we have

Proposition 4.2. *Assume that $\underline{x}(0) \cap \bar{X}^+ = \{p_1, p_2\}$, and $\underline{x}(0) \cap \bar{X}^- \neq \emptyset$. Then for a generic J_0 , the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0 . Given an element $\bar{f} : \bar{C} \rightarrow X_{\sharp}$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover,*

(1) C_+ is irreducible and $\bar{f}(C_+)$ realizes the class $[H]$ passing through $\{p_1, p_2\}$. The curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{\delta_1, 0}(p^!d -$

$[E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)$ for some $q \in V$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.

(2) C_+ has exactly two irreducible components and the image of each component realizes the class $[H]$ and passing through one point of $\{p_1, p_2\}$. The curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 2\delta_1}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.

Proof. Example 11.4 and Lemma 14.6 in [14] implies that no component of \bar{C} is entirely mapped into V . In the real blow-up $\bar{X}^- \cong X_{1,0}$, $[E]^2 = -1$. The adjunction formula implies that $c_1(\bar{X}^-) \cdot [E] = 1$. Suppose $\bar{f}_*[C_+] = a[H]$, $\bar{f}_*[C_-] = p^!d - b[E]$. Thus by considering a representative of V in \bar{X}^+ and another in \bar{X}^- respectively, we have

$$a = \bar{f}_*[C_+] \cdot [H] = (p^!d - b[E]) \cdot [E] = p^!d \cdot [E] + b = b.$$

Since $\underline{x}(0) \cap \bar{X}^+ = \{p_1, p_2\}$, so $\bar{f}(C_+)$ passes through the two points p_1, p_2 . Then $\underline{x}(0) \cap \bar{X}^- \neq \emptyset$ implies that $a = b \geq 1$ and $c_1(X) \cdot d \geq 4$. Therefore, $\bar{f}(C_-)$ passes through all the $c_1(X) \cdot d - 3$ points in $\underline{x}(0) \cap \bar{X}^-$ and realizes the class $p^!d - b[E]$ in $H_2(\bar{X}^-; \mathbb{Z})$.

Suppose that C_- consists of irreducible components $\{C_{-i}\}_{i=1}^m$ with $0 \leq k \leq m$ irreducible components $\{C_{-i}\}_{i=1}^k$ such that the restriction $\bar{f}|_{C_{-i}}$, $i = 1, \dots, k$, is non-simple which factors through a non-trivial ramified covering of degree $\delta_i \geq 2$ of a simple map $f_i : \mathbb{P}^1 \rightarrow \bar{X}^-$. Assume that $(f_i)_*[\mathbb{P}^1] = d_i$, $i = 1, \dots, k$, and $\bar{f}_*[C_{-j}] = d_j$, $j = k+1, \dots, m$. Then $\sum_{i=1}^k \delta_i d_i + \sum_{j=k+1}^m d_j = p^!d - b[E]$.

$$c_1(\bar{X}^-) \cdot \left(\sum_{i=1}^k d_i \right) - k + c_1(\bar{X}^-) \cdot \left(\sum_{j=k+1}^m d_j \right) - (m - k)$$

$$\geq c_1(X) \cdot d - 3$$

$$= c_1(\bar{X}^-) \cdot \left(\sum_{i=1}^k \delta_i d_i + \sum_{j=k+1}^m d_j \right) + b - 3.$$

Therefore,

$$(18) \quad \sum_{i=1}^k (1 - \delta_i) c_1(\bar{X}^-) \cdot d_i \geq m + b - 3.$$

Since $c_1(\bar{X}^-) \cdot d_i \geq 0$, so we have

$$m + b \leq 3.$$

First of all, we assume that $k \geq 1$. Then (18) implies $m + b < 3$. Thus we have $m = b = 1$. This implies that C_- has one component. Furthermore,

assume that $\bar{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $\bar{f}_- : \mathbb{P}^1 \rightarrow \bar{X}^-$. Then we have

$$\frac{1}{\delta} c_1(\bar{X}^-) \cdot (p^!d - b[E]) - 1 \geq c_1(X) \cdot d - 3.$$

Therefore,

$$c_1(X) \cdot d + 2\delta - \delta c_1(X) \cdot d \geq b.$$

Since $\delta \geq 2$ and $c_1(X) \cdot d \geq 4$. So $b \leq 0$. This is in contradiction with $b = 1$. This contradiction implies that $k = 0$.

Next, we assume that $k = 0$. (18) implies that we only need to consider the following two cases.

Case I: $m = 2, b = 1$.

In this case, $\bar{f}|_{C_+}$ is constrained by $\{p_1, p_2\}$ and $\bar{f}|_{C_+} \in \mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$. Therefore $\bar{f}|_{C_-}$ has to pass the point of intersection of $\bar{f}|_{C_+}$ and V which is distinct to $\underline{x}(0) \cap \bar{X}^-$. $\bar{f}|_{C_-}$ will pass $c_1(X) \cdot d - 2 = c_1(\bar{X}) \cdot (p^!d - [E]) - 1$ distinct points which implies that $\bar{f}|_{C_-}$ is irreducible. This is in contradiction with that C_- has two components. This contradiction implies that this case is impossible.

Case II: $m = 1, b = 1$ or 2 .

If $b = a = 1$, C_+ must have exactly 1 component and its image under \bar{f} realizes the class $[H]$. $\bar{f}|_{C_+}$ is simple because of Lemma 2.5. $\bar{f}|_{C_+} \in \mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$. By the positivity of intersection, there only has one curve in $\mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$ which is an embedded simple curve. Denote by q the point of intersection of $\bar{f}|_{C_+}$ and V . The point q depends only on $\mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$. Therefore $\bar{f}|_{C_-}$ has to pass $\underline{x}(0) \cap \bar{X}^- \sqcup \{q\}$, and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{\delta_1, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)$.

If $b = a = 2$, $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 2\delta_1}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. $\bar{f}|_{C_-}$ intersects E transversely in 2 distinct points. Note that the curve \bar{C} is rational and any component of $\bar{f}(C_+)$ intersects E in \bar{X}^+ , so C_+ has exactly 2 irreducible components. Furthermore each component of $\bar{f}(C_+)$ realizes $[H]$ and passes through one point of $\{p_1, p_2\}$.

The rest of the Proposition can be proved similar to Proposition 3.1. We omit it here. \square

Proposition 4.3. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d \geq 4$ and $\tau_*d = -d$. Denote by $p : X_{1,0} \rightarrow X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Then if $s \geq 1$, we have*

$$\begin{aligned} W_{X_{\mathbb{R}}}(d, s-1) &= \sum_{C_1 \in \mathbb{R}\mathcal{C}^{\delta_1^T, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)} (-1)^{m_{X_{1,0}}(C_1)} \\ (19) \quad &+ 2 \sum_{C_2 \in \mathbb{R}\mathcal{C}^{0, 2\delta_1^T}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_2)}, \end{aligned}$$

$$\begin{aligned}
W_{X_{\mathbb{R}}}(d, s) &= \sum_{C_1 \in \mathbb{R}\mathcal{C}^{\delta_1^T, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)} (-1)^{m_{X_{1,0}}(C_1)} \\
(20) \quad &- 2 \sum_{C_2 \in \mathbb{R}\mathcal{C}^{0, 2\delta_1^c}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_2)},
\end{aligned}$$

where E is the exceptional divisor and q is some particular point in V .

Proof. Equip the small disc Δ with the standard complex conjugation. From subsection 2.3, we know one can equip the symplectic sum $\pi : \mathcal{Z} \rightarrow \Delta$ with a real structure $\tau_{\mathcal{Z}}$ which is induced by the real structures τ_-, τ_+ on the real symplectic cuts \bar{X}^- and \bar{X}^+ such that the map $\pi : \mathcal{Z} \rightarrow \Delta$ is real. Choose a set of real sections $\underline{x} : \Delta \rightarrow \mathcal{Z}$ such that $\underline{x}(0) \cap \bar{X}_{1,0}^+$ contains two points. Let $\bar{f} : \bar{C} \rightarrow X_{\sharp}$ be a real element in $\mathbb{R}\mathcal{C}(d, \underline{x}(0), J_0)$.

Next, we will divide the proof into two cases according to the type of real configuration points.

Case I; $\underline{x}(0) \cap \bar{X}^+ = \{p_1, p_2\}$ and $p_1, p_2 \in \mathbb{R}X$.

From Proposition 4.2, there are two types of the limited curve \bar{f} as following.

Type I-1: C_+ has only one component.

$f_*[C_+] = [H]$ and $\bar{f}|_{C_+} \in \mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$ is an embedded simple curve. The intersection point q of $\bar{f}(C_+)$ with V , determined by $\mathcal{C}^{0, \delta_1}([H], \{p_1, p_2\}, J_0)$, has to be real. In this case, since $\bar{f}(C_+)$ has no self-intersection point, so $\bar{f}(C_+)$ has no node. Therefore, there is only one possibility to recover a real curve $\bar{f}(\bar{C})$ from $\bar{f}|_{C_+}$ when $\bar{f}|_{C_-}$ is fixed. So we have

$$\begin{aligned}
m_{X_{\sharp}}(\bar{f}(\bar{C})) &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}) + m_{\bar{X}^+}(\bar{f}|_{C_+}) \\
(21) \quad &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}).
\end{aligned}$$

Type I-2: C_+ has exactly two irreducible components C_{+i} , $i = 1, 2$. In this case, $\bar{f}_*[C_{+i}] = [H]$ and $\bar{f}|_{C_{+i}}$ is an embedded simple curve. By the positivity of intersections, $\bar{f}(C_{+1})$ intersects $\bar{f}(C_{+2})$ at one point. This point has to be a real node of $\bar{f}(C_+)$ because $\bar{f}(C_+)$ is real. Since $\bar{f}(C_{+i})$ passes $p_i \in \mathbb{R}X$, $\bar{f}(C_{+1})$ and $\bar{f}(C_{+2})$ can not be two τ -conjugated components. Therefore the real nodal point of $\bar{f}(C_+)$ has to be non-isolated. Moreover, each $\bar{f}(C_{+i})$ intersects V at a real point q_i transversally and $\bar{f}|_{C_{+i}} \in \mathcal{C}^{\delta_1^T, 0}([H], \{p_i\} \sqcup \{q_i\}, J_0)$. From Proposition 4.2 (2), we can get the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 2\delta_1}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. $\bar{f}|_{C_+}$ and $\bar{f}|_{C_-}$ form the limited curve \bar{f} . We know $\bar{f}(C_-)$ intersects V at two real non-prescribed points transversally. Therefore $\bar{f}|_{C_-} \in \mathcal{C}^{0, 2\delta_1^T}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. Therefore, there are two possibilities to recover a real

curve $\bar{f}(\bar{C})$ from $\bar{f}|_{C_+}$ when $\bar{f}|_{C_-}$ is fixed. We have

$$\begin{aligned} m_{X_{\sharp}}(\bar{f}(\bar{C})) &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}) + m_{\bar{X}^+}(\bar{f}|_{C_+}) \\ (22) \qquad \qquad \qquad &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}). \end{aligned}$$

By Proposition 4.2, an element \bar{f} of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0. So the latter has to be real when \bar{f} is real and $\lambda \in \mathbb{C}^*$ is small. When deforming \bar{f} , no node appears in a neighborhood of $V \cap \bar{f}(\bar{C})$. From the analysis of Case I, we know the elements of $\mathcal{C}(d, \underline{x}(0), J_0)$ have two different types. Therefore the elements of $\mathbb{R}\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ will degenerate into two types: in type I-1, an element of $\mathbb{R}\mathcal{C}^{\delta_1^r, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)$ corresponds to a unique element of the limited curve; in type I-2, an element of $\mathbb{R}\mathcal{C}^{0, 2\delta_1^r}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$ corresponds to two limited curves with the same mass. One can get formula (19) easily from (21) and (22).

Case II: $\underline{x}(0) \cap X^+ = \{p, p'\}$ with $p, p' \in X \setminus \mathbb{R}X$ and $\tau(p_1) = p_2$.

From Proposition 4.2, we also obtain two types of the limited curve \bar{f} as following.

Type II-1: C_+ has only one component.

$\bar{f}_*[C_+] = [H]$ and $\bar{f}|_{C_+} \in \mathcal{C}^{0, \delta_1}([H], \{p, p'\}, J_0)$ is an embedded simple curve. The τ -conjugated pair $p, p' \in X \setminus \mathbb{R}X$ can be chosen such that the intersection point, determined by $\mathcal{C}^{0, \delta_1}([H], \{p, p'\}, J_0)$, is also q . So $\bar{f}|_{C_-}$ belongs to $\mathcal{C}^{\delta_1^r, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)$, and the remaining argument is the same as that in the case I. We omit it here.

Type II-2: C_+ has exactly two irreducible components $C_{+i}, i = 1, 2$. In this case, $\bar{f}_*[C_{+i}] = [H]$ and $\bar{f}|_{C_{+i}}$ is an embedded simple curve. By the positivity of intersections, $\bar{f}(C_{+1})$ intersects $\bar{f}(C_{+2})$ at one real point which is a real node of $\bar{f}(C_+)$. Since $\bar{f}(C_{+i})$ passes $p_i \in X \setminus \mathbb{R}X$ with $\tau(p_1) = p_2$, $\bar{f}(C_{+1})$ and $\bar{f}(C_{+2})$ are two τ -conjugated components. Therefore the real nodal point of $\bar{f}(C_+)$ has to be isolated. Moreover, each $\bar{f}(C_{+i})$ intersects V at a point q_i transversally with $\tau(q_1) = q_2$. From Proposition 4.2 (2), we can get the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 2\delta_1}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. $\bar{f}|_{C_+}$ and $\bar{f}|_{C_-}$ form the limited curve \bar{f} . We know $\bar{f}(C_-)$ intersects V at two τ -conjugated non-prescribed points transversally. Therefore $\bar{f}|_{C_-} \in \mathcal{C}^{0, 2\delta_1^c}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$. There are two possibilities to recover a real curve $\bar{f}(\bar{C})$ from $\bar{f}|_{C_+}$ when $\bar{f}|_{C_-}$ is fixed. We have

$$\begin{aligned} m_{X_{\sharp}}(\bar{f}(\bar{C})) &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}) + m_{\bar{X}^+}(\bar{f}|_{C_+}) \\ (23) \qquad \qquad \qquad &= m_{\bar{X}^-}(\bar{f}|_{\bar{C}_-}) + 1. \end{aligned}$$

By Proposition 4.2, an element \bar{f} of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0. So the latter has to be real when

\bar{f} is real and $\lambda \in \mathbb{C}^*$ is small. When deforming \bar{f} , no node appears in a neighborhood of $V \cap \bar{f}(\bar{C})$. From the analysis of Case II, we know the elements of $\mathcal{C}(d, \underline{x}(0), J_0)$ have two different types. Therefore the elements of $\mathbb{R}\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ will degenerate into two types: in type II-1, an element of $\mathbb{R}\mathcal{C}^{\delta_1^r, 0}(p^!d - [E], \underline{x}(0) \cap \bar{X}^- \sqcup \{q\}, J_0)$ corresponds to a unique element of the limited curve; in type II-2, an element of $\mathbb{R}\mathcal{C}^{0, 2\delta_1^c}(p^!d - 2[E], \underline{x}(0) \cap \bar{X}^-, J_0)$ corresponds to two limited curves with the same mass. One can get formula (20) easily from (21) and (23). \square

Next, perform the real symplectic cut along the exceptional divisor E in $X_{1,0}$. We can get $\bar{X}_{1,0}^+ \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$, $\bar{X}_{1,0}^- \cong X_{1,0}$, V as in subsection 3.1.

Let $\tilde{\mathcal{Z}}$ be the real symplectic sum of $\bar{X}_{1,0}^+$ and $\bar{X}_{1,0}^-$ along V . Let $p^!d - 2[E] \in H_2(\tilde{\mathcal{Z}}_\lambda; \mathbb{Z})$, where $d \in H_2(X; \mathbb{Z})$. Choose $\tilde{\underline{x}}(\lambda)$ a set of $c_1(X) \cdot d - 3$ real sections $\Delta \rightarrow \tilde{\mathcal{Z}}$ such that $\tilde{\underline{x}}(0) \cap V = \emptyset$, $\tilde{\underline{x}}(0) \cap \bar{X}_{1,0}^+ = \emptyset$. Choose an almost complex structure \tilde{J} on $\tilde{\mathcal{Z}}$ as before. Denote \tilde{X}_\sharp , $\mathcal{C}(p^!d - 2[E], \tilde{\underline{x}}(0), \tilde{J}_0)$, C_* , $* = +, -$, as in subsection 3.1.

Proposition 4.4. *For a generic \tilde{J}_0 , the set $\mathcal{C}(p^!d - 2[E], \tilde{\underline{x}}(0), \tilde{J}_0)$ is finite, and only depends on $\tilde{\underline{x}}(0)$ and \tilde{J}_0 . Given an element $\bar{f} : \bar{C} \rightarrow \tilde{X}_\sharp$ of $\mathcal{C}(p^!d - 2[E], \tilde{\underline{x}}(0), \tilde{J}_0)$, the restriction of \bar{f} to any component of \bar{C} is a simple map, and no irreducible component of \bar{C} is entirely mapped into V . Moreover, the curve C_- is irreducible and $\bar{f}|_{C_-}$ is an element of $\mathcal{C}^{0, 2\delta_1}(p^!d - 2[E], \tilde{\underline{x}}(0) \cap \bar{X}_{1,0}^-, \tilde{J}_0)$. The curve C_+ has two irreducible components. Each component of $\bar{f}(C_+)$ realizes the fiber class $[F]$ in $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$. The map \bar{f} is the limit of a unique element of $\mathcal{C}(p^!d - 2[E], \tilde{\underline{x}}(\lambda), \tilde{J}_\lambda)$ as λ goes to 0.*

Proof. As before, we know that no component of \bar{C} is entirely mapped into V . Since $\bar{f}_*[\bar{C}] = p^!d - 2[E]$, we may suppose $\bar{f}_*[C_+] = a[F] + b[E_\infty]$, $a, b \geq 0$, $\bar{f}_*[C_-] = p^!d - k[E]$, $k \geq 0$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow E$ with $F \cdot [E_0] = 1$ and $F \cdot [E_\infty] = 1$. Then

$$a + b = (a[F] + b[E_\infty]) \cdot [E_\infty] = (p^!d - k[E]) \cdot [E] = k,$$

$$a = (a[F] + b[E_\infty]) \cdot [E_0] = (p^!d - 2[E]) \cdot [E] = 2.$$

This implies $k \geq 2$.

In $\bar{X}_{1,0}^-$, we know that $\bar{f}|_{C_-}$ passes through

$$c_1(X) \cdot d - 3 = c_1(\bar{X}_{1,0}^-) \cdot (p^!d - 2[E]) - 1$$

distinct points. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

Assume that $\bar{f}|_{C_-}$ is non-simple. Then $\bar{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{P}^1 \rightarrow \bar{X}^-$. Then

$(f_0)_*[\mathbb{P}^1] = \frac{1}{\delta}(p^!d - k[E])$. Therefore,

$$\frac{1}{\delta}c_1(\tilde{X}^-) \cdot (p^!d - k[E]) - 1 \geq c_1(X) \cdot d - 3.$$

This implies

$$c_1(X) \cdot d - \delta c_1(X) \cdot d + 2\delta \geq k.$$

Since $\delta \geq 2$, $c_1(X) \cdot d \geq 4$, so we have $k \leq 0$. This is in contradiction with $k \geq 2$. Thus $\tilde{f}|_{C_-}$ is simple.

On the other hand, we have

$$\begin{aligned} c_1(\tilde{X}^-) \cdot (p^!d - k[E]) - 1 &= c_1(X) \cdot d - k - 1 \\ &\geq c_1(X) \cdot d - 3. \end{aligned}$$

This implies $k \leq 2$. Therefore, we have $k = 2$, $b = 0$. Since the image of C_- under \tilde{f} intersects V transversally in 2 distinct points. Therefore, C_+ has two irreducible components C_{+i} such that $\tilde{f}_*[C_{+i}] = [F]$, $i = 1, 2$.

The rest of the Proposition can be obtained by the similar argument in the proof of Proposition 3.1. We omit it here. \square

Proposition 4.5. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d \geq 4$ and $\tau_*d = -d$. Denote by $p : X_{1,0} \rightarrow X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Then if $s \geq 1$,*

$$\begin{aligned} W_{X_{1,0}}(p^!(d) - 2[E], s - 1) &= \sum_{C_- \in \mathbb{R}C^{0,2\delta_1^T}(p^!d - 2[E], \tilde{\mathcal{X}}_3(0) \cap \tilde{X}^-, \tilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)} \\ &+ \sum_{C_- \in \mathbb{R}C^{0,2\delta_1^c}(p^!d - 2[E], \tilde{\mathcal{X}}_3(0) \cap \tilde{X}^-, \tilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)}, \end{aligned}$$

where E is the exceptional divisor.

Remark 4.6. The proof of Proposition 4.5 is similar to Proposition 3.2. Proposition 4.3 and Proposition 4.5 imply Theorem 1.4.

4.2. Generating function. In this subsection, we restate our formulae in the form of generating functions. Denote by

$$W_{X_{\mathbb{R}}, L, F}^d(T) = \sum_{s=0}^{\lfloor \frac{c_1(X) \cdot d - 1}{2} \rfloor} W_{X_{\mathbb{R}}, L, F}(d, s) T^s \in \mathbb{Z}[T]$$

the generating function of Welschinger invariants which encodes all the information of the Welschinger invariants.

Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold. If $\mathbb{R}X$ is disconnected, the previous formulae are still true, and can be proved in the same method as Theorem 1.4 and Corollary 1.3. Suppose $\underline{x}' \subset X$ is a real set consisting of r' points in L and s' pairs of τ -conjugated points in X with $r' + 2s' \leq c_1(X) \cdot d - 1$. We denote the connected component of $\mathbb{R}X_{r', s'}$ corresponding to L by \tilde{L} . If there is only one blown-up real point in L , $\tilde{L} = L \# \mathbb{R}P^2$. We assume F has a τ -invariant compact representative $\mathcal{F} \subset X \setminus \underline{x}'$, and denote

$\tilde{F} = p^!F$. Denote by $p : X_{r',s'} \rightarrow X$ the projection of the real symplectic blow-up of X at \underline{x}' . Then

$$\begin{aligned} W_{X_{\mathbb{R}},L,F}^d(T) &= W_{X_{r',s'},\tilde{L},\tilde{F}}^{p^!d}(T), \\ W_{X_{\mathbb{R}},L,F}^d(T) - W_{X_{\mathbb{R}},L,F}(d,0) - \dots - W_{X_{\mathbb{R}},L,F}(d,s'-1)T^{s'-1} \\ &= W_{X_{r',s'},\tilde{L},\tilde{F}}^{p^!d - \sum_{i=1}^{r'}[E_i] - \sum_{j=1}^{s'}([E'_j] + [E''_j])}(T)T^{s'}, \\ W_{X_{\mathbb{R}},L,F}^d(T)T &= W_{X_{\mathbb{R}},L,F}^d(T) - W_{X_{\mathbb{R}},L,F}(d,0) + 2W_{X_{1,0},\tilde{L},\tilde{F}}^{p^!d-2[E]}(T)T, \end{aligned}$$

where E_i , E'_j , E''_j denote the exceptional divisors corresponding to the real set \underline{x}' respectively.

5. REAL ENUMERATION

In this section we will give some applications of the blow-up formula of Welschinger invariants.

5.1. Blow-up of \mathbb{CP}^2 . Let $\mathbb{CP}_{r,s}^2$ denote the blow-up of \mathbb{CP}^2 at r real points and s pairs of conjugated points. The projective plane with the standard symplectic structure and complex conjugate is a real symplectic manifold. [1] proved a recursive formula of Welschinger invariants in the projective plane. Using the results of [1] and the blow-up formula of Welschinger invariants (Corollary 1.3), we can compute the invariants of $\mathbb{CP}_{r,s}^2$.

s	0	1	2	3	4
$W(c_1(X), 0)$	8	6	4	2	0
$W(c_1(X), 1)$	6	4	2	0	-
$W(c_1(X), 2)$	4	2	0	-	-
$W(c_1(X), 3)$	2	0	-	-	-

	$W([H], 0)$	$W(2[H], 0)$	$W(4[H], 0)$
$X = \mathbb{CP}_{3,0}^2$	1	1	240
$X = \mathbb{CP}_{1,1}^2$	1	1	144

Table 1. Welschinger invariants of $\mathbb{CP}_{r,s}^2$ with $r + 2s \leq 8$

Note that the Welschinger invariants of $\mathbb{CP}_{r,s}^2$ with purely real point constraints were computed in [15, 16, 17, 19, 24, 20, 21]. And the Welschinger invariants of $\mathbb{CP}_{r,s}^2$ were totally computed in [10]. The invariants for $r+2s \leq 3$ with arbitrary real and conjugated pairs of point constraints were studied by [5]. For $6 \leq r + 2s \leq 8$, $W_{\mathbb{CP}_{r,s}^2}(d, s')$ were considered in [2] with any point constraints.

5.2. Conic bundles and Del Pezzo surfaces of degree 2. Recall that there are 12 topological types of degree 2 real Del Pezzo surfaces. I. Itenberg, V. Kharlamov, and E. Shustin [24] computed the Welschinger invariants with purely real point constraints of all degree 2 real Del Pezzo surfaces. E. Brugallé [2], computed the Welschinger invariants with arbitrary point constraints of real degree 2 Del Pezzo surfaces with a non-orientable real part. A. Horev and J. Solomon [10] also computed Welschinger invariants with arbitrary point constraints of some degree 2 Del Pezzo surfaces with a non-orientable real part. E. Brugallé [2, 3] computed the Welschinger invariants of all the real degree 1 Del Pezzo surface. Since every degree $2n - 1$ Del Pezzo surface which is not the minimal Del Pezzo surface is the blow-up of a degree $2n$ Del Pezzo surface at a real point. We can use the blow-up formula to compute the Welschinger invariants with conjugated point constraints in the remaining five topological types of degree 2 real Del Pezzo surfaces with no non-orientable real part.

Let \mathbb{B}^n be the real conic bundle with $2n$ singular fibers and X^1 be the minimal real del Pezzo surface of degree 2. Endow $\mathbb{P}^1 \times \mathbb{P}^1$ with the standard real structure. So $X^1, \mathbb{B}^3, \mathbb{B}_{0,1}^2, \mathbb{B}_{0,2}^1, (\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$ are the five topological types of real Del Pezzo surfaces of degree 2 with real parts $\sqcup 4S^2 \sqcup 3S^2, \sqcup 2S^2, S^2, S^1 \times S^1$, respectively. The following tables are taken from [2, 3, 24].

	X^1	\mathbb{B}^3	$\mathbb{B}_{0,1}^2$	$\mathbb{B}_{0,2}^1$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$
$W(2c_1(X), 0)$	0	0	0	8	32

	$X_{1,0}^1$	$\mathbb{B}_{1,0}^3$	$\mathbb{B}_{1,1}^2$	$\mathbb{B}_{1,2}^1$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{1,3}$
$W(2c_1(X), 0)$	18	10	6	6	10

From Welschinger's wall-crossing formula:

$$W_{X_{\mathbb{R}}}(d, s - 1) = W_{X_{\mathbb{R}}}(d, s) + 2W_{X_{1,0}}(p^1 d - 2[E], s - 1),$$

we can obtain the following values.

	X^1	\mathbb{B}^3	$\mathbb{B}_{0,1}^2$	$\mathbb{B}_{0,2}^1$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$
$W(2c_1(X), 1)$	-36	-20	-12	-4	12

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